Realization of Lindahl Equilibria for the Bergstrom-Cornes Environment

Masahiro Watabe

Abstract

The paper considers the allocative efficiency and the informational aspects for public goods economies with the Bergstrom-Cornes form of utility functions $u_i(x_i, y) = \gamma(y)x_i + v_i(y)$, where x_i is the amount of the private good consumed by consumer i and y is the vector of public goods. The informationally decentralized resource allocation process presented here fully realizes Lindahl allocations.

Keywords: Mechanism Design · Lindahl equilibria · Independence of income distribution

1 Introduction

This paper examines the design of informationally decentralized resource allocation process satisfying Pareto optimality for public goods economies. Consumers' preferences are represented by a special form of utility functions introduced by Bergstrom and Cornes (1983). The purpose of this paper is to construct a resource allocation process which attains Lindahl allocations through communication among consumers and the government in a decentralized way.

The concept of Lindahl equilibrium has played a crucial role in the literature on public goods theory. It is important to achieve Lindahl allocations because these result in Pareto optimal and individually rational allocations. A traditional interpretation of Lindahl equilibrium can be found in Hurwicz (1986):

> In one scenario representing the Lindahl mechanism, there is an "auctioneer" proposing shares p_i $(i = 1, \cdots, n)$ of public the aggregate cost of a public service to be borne by the n participants; in turn, the ith participant responds by specifying the level y_i of the public service that would maximize his or her utility given p_i; agent i would then contribute $p_i y_i$ to cover the costs of public service. Equilibrium obtains when the shares p_i are so chosen that all agents desire the same level of public service, that is, $y_1 = \cdots = y_n$.

In other words, the Lindahl equilibrium concept is one in which each consumer would behave as a price taker. However, it is difficult to believe such price taking behavior in the personalized market for public goods. The concept of Lindahl equilibrium is relevant to private-ownership economies, in the sense that the public goods are traded in markets and the government gives profit maximizing firms an exclusive franchise to produce public goods. The present paper, in contrast to the above scenario, attempts to illustrate the Lindahl equilibrium with the government as an auctioneer who proposes an amount of public goods as planning instruments rather than personalized prices.

I focus on the optimal level of provision of public goods without any distributional concerns. It is well-known that separation from allocational decisions from distributional decisions is possible when preferences of all consumers are quasi-linear. However, quasi-linearity implies a zero income elasticity of demand for public goods. Bergstrom and Cornes (1983) point out that this is not consistent with empirical findings. They find a class of analytically tractable utility functions, broader than the class of quasi-linear utility functions, for which separation of allocation from distribution is possible. I assume that consumers' preferences are represented by the Bergstrom-Cornes form of utility functions.

The rest of the present paper is organized as follows. In Section 2, the Bergstrom-Cornes environment. The Pareto and the Lindahl choice rules are defined. In Section 3, a concept of resource allocation process and a few properties of it are defined. In Section 4, it is shown that there exists an informationally decentralized resource allocation process which fully realizes the Lindahl choice rule (Theorem 1). The set of equilibrium allocations of the informationally decentralized resource allocation process coincides the set of Lindahl allocations exactly. Finally, I give some concluding remarks in Section 5.

2 Bergstrom-Cornes Economies with Convex Cost Functions

I consider economies with m public goods, one private good as a numeraire, n consumers, and one producer. The set of public goods, the set of commodities, and the set of consumers are denoted by $K = \{1, \cdots, m\}$, $H = \{1, \cdots, m+1\}$, and $I = \{1, \cdots, n\}$, respectively. The production possibility set is assumed to be represented by a twice-continuously differentiable and strictly convex cost function $C : \mathbb{R}^K_+ \to \mathbb{R}_+$ that satisfies strict differentiable monotonicity, and $C(0_K) = 0_K$.[1] The input requirement of the numeraire for y is given as $C(y)$. The set of admissible cost functions is denoted by $\mathcal{C}$.

Preference relations are represented by a twice-continuously differentiable and quasi-concave utility function which satisfies strict differentiable monotonicity. The set of admissible utility functions is denoted by $\mathcal{U}$. Bergstrom and Cornes (1983) formulate a particular subset of $\mathcal{U}$. Let γ be a twice continuously differentiable function of $\mathbb{R}^K_+$ into $\mathbb{R}_{++}$. For each consumer i, define

$$\mathcal{U}_i^\gamma = \{u^\gamma \in \mathcal{U} \mid \text{There is a real-valued function } v_i \text{ on } \mathbb{R}^K_+ \text{ such that } u_i^\gamma(x_i, y) = \gamma(y)x_i + v_i(y) \text{ for every } (x_i, y) \in \mathbb{R}^H_+\}.$$

I put the following additional assumptions on preferences.

Assumption 1. For every consumer i, $u_i^\gamma(x_i, y) > u_i^\gamma(\hat{x}_i, \hat{y})$ for all $(x_i, y) \gg 0_H$ and all $(\hat{x}_i, \hat{y}) \in \partial\mathbb{R}^H_+$, where $\partial\mathbb{R}^H_+$ is the boundary of $\mathbb{R}^H_+$.

Assumption 2. For every consumer i, each element $u_i^\gamma \in \mathcal{U}_i^\gamma$ satisfies the *differential strict quasi-concavity*, which means that

[1] In what follows, the null vector in the $\#A$-dimensional Euclidean space is denoted by $0_{\#A}$.

$zD^2u_i^\gamma(x_i,y)\cdot z<0$ for every $(x_i,y)\in\mathbb{R}^H_{++}$ and $z\in\mathbb{R}^H\setminus\{0_H\}$ such that $\nabla u_i^\gamma(x_i,y)\cdot z=0$.[23]

Each consumer i has an endowment $w_i\in\mathbb{R}_{++}$ of the private good, but no endowments of public goods. Let $\bar{w}=\sum_{i\in I}w_i$ be the aggregate endowment of the private good. For each consumer i, I denote by E_i^γ the class of characteristics of consumer i, that is, $E_i^\gamma=\mathbb{R}^H_{++}\times\mathcal{U}_i^\gamma$. A generic element of E_i^γ is denoted by $e_i=(w_i,u_i^\gamma)$, where $u_i^\gamma(x_i,y)=\gamma(y)x_i+v_i(y)$ for each element (x_i,y) of $\mathbb{R}^H_+$.

I denote by E_J the class of characteristics of the producer, that is, $E_J=\mathcal{C}$. A generic element of E_J is denoted by $e_J=C$. In what follows, J stands for the index of the producer.

An $(n+1)-$tuple $e=((e_i)_{i\in I},e_J)$, where $e_i\in E_i$ for every consumer i and $e_J\in E_J$ is called an *environment*. The *Bergstrom-Cornes environment* is defined by $E^\gamma=\prod_{i\in I}E_i^\gamma\times E_J$.

Let $e=((\omega_i,u_i^\gamma)_{i\in I},C)$ be any element of E^γ. An $m-$tuple $y=(y_k)_{k\in K}$ is called a *production of public goods*. For each consumer i, his *consumption* is denoted by an $(m+1)-$tuple (x_i,y). An $(n+m)-$tuple $((x_i)_{i\in I},y)$ is called an *allocation for* e. The *set of feasible allocations for* e is defined by

$$\mathcal{Z}(e)=\{((x_i)_{i\in I},y)\in\mathbb{R}^I_+\times\mathbb{R}^K_+\mid \textstyle\sum_{i\in I}x_i+C(y)\leqslant\sum_{i\in I}w_i\}.$$

Now, I define a criteria of social desirability. A relation $\mathcal{O}$ of E^γ into $\mathbb{R}^I\times\mathbb{R}^K$ is called a *choice rule* if $\mathcal{O}(e)\subseteq\mathcal{Z}(e)$ for every element e of E^γ. An element $((x_i^*)_{\in I},y^*)$ of $\mathcal{Z}(e)$ is said to *be Pareto optimal for* e if there is no element $((x_i)_{\in I},y)$ of $\mathcal{Z}(e)$ such that (1) $u_i^\gamma(x_i,y)\geqslant u_i^\gamma(x_i^*,y^*)$ for every consumer i, and (2) $u_i^\gamma(x_i,y)>u_i^\gamma(x_i^*,y^*)$ for some consumer i. Denote by $PT(e)$ the set of such allocations. The relation PT of E^γ into $\mathbb{R}^I\times\mathbb{R}^K$ is referred to as the *Pareto choice rule*. Much of literature on public goods provision has focused on Pareto optimality as a basic social goal.

Because of the corresponding production technologies exhibit decreasing returns to scale, the maximized profit is non-negative. Each consumer i has a share $\theta_i\in[0,1]$ on the profit with $\sum_{i\in I}\theta_i=1$. For each $q_i\in\mathbb{R}^K_{++}$ and $\pi\in\mathbb{R}$, define the budget correspondence by

$$\gamma_i(q_i,\pi,e_i)=\{(x_i,y)\in\mathbb{R}^H_+\mid x_i+q_i\cdot y\leqslant w_i+\theta_i\pi\},$$

and the demand correspondence by

$$D_i(q_i,\pi,e_i)=\{(x_i^*,y^*)\in\gamma_i(q_i,\pi,e_i)\mid u_i^\gamma(x_i^*,y^*)\geqslant u_i^\gamma(x_i,y)\text{ for every }(x_i,y)\in\gamma_i(q_i,\pi,e_i)\}$$

for each consumer i. An element $((x_i^*)_{\in I},y^*)\in\mathcal{Z}(e)$ is called a *Lindahl allocation for e with respect to* θ if there exists $q=(q_i)_{i\in I}\in(\mathbb{R}^K_{++})^I$ with $\sum_{i\in I}q_i=p$ and $p\cdot y^*-C(y^*)=\max\{p\cdot y-C(y)\mid y\in\mathbb{R}^K_+\}$ such that $(x_i^*,y^*)\in D_i(q_i,p\cdot y^*-C(y^*),e_i)$ for every consumer i. Denote by $LD_\theta(e)$ the set of such allocations. The relation LD_θ of E^γ into $\mathbb{R}^I\times\mathbb{R}^K$ is referred to as the *Lindahl choice rule with respect to* θ. See Foley (1967) and Milleron (1972), for instance, for the early literature on public goods theory.

[2] The differential strict quasi-concavity is sufficient for the strict quasi-concavity, which means the following: for every $(x_i,y),(x_i',y')\in\mathbb{R}^H_+$, if $(x_i,y)\neq(x_i',y')$ and $u_i^\gamma(x_i,y)\geqslant u_i^\gamma(x_i',y')$, then $u_i^\gamma((1-t)x_i+tx_i',(1-t)y+ty')>u_i^\gamma(x_i',y')$ for every $t\in]0,1[$.

[3] Throughout the present paper, I always think of the gradient as a row vector.

I present an example to illustrate the notion of Lindahl equilibria for the Bergstrom-Cornes environment.

Example 1. Consider the environment with one public good. The strictly convex cost function is written as $C(y) = 4y^2$. There are two consumers. Consumer i has preferences are represented as $u_i^\gamma(x_i, y) = (y + \varepsilon)x_i + \alpha_i y - \frac{1}{2}y^2$, where $\varepsilon > 0$ and $\alpha_i > 0$. If the coefficient α_i is large enough, it is shown that $u_i^\gamma \in \mathcal{U}_i^\gamma$. Notice that $\partial u_i^\gamma(x_i, y)/\partial x_i = y+\varepsilon > 0$ and $\partial u_i^\gamma(x_i, y)/\partial y = x_i+\alpha_i-y > 0$, provided that $\alpha_i > 0$ is sufficiently large. Let $z = (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that $\nabla u_i^\gamma(x_i, y) \cdot z = 0$. The latter expression ensures that a and b have opposite signs because of the strict differentiable monotonicity. If $b = 0$, then it must be the case that $a \times (\partial u_i^\gamma(x_i, y)/\partial x_i) = a(y+\varepsilon) = 0$, which implies $a = 0$. This is a contradiction. Therefore, $b \neq 0$. Then, $zD^2u_i^\gamma(x_i, y) \cdot z = 2ab - b^2 < 0$ for sure because $ab < 0$. Therefore, utility functions of the form $u_i^\gamma(x_i, y) = (y+\varepsilon)x_i+\alpha_i y-\frac{1}{2}y^2$ are differentially strictly quasi-concave.

Let (x_1, x_2, y) be an interior Lindahl equilibrium allocation. Each consumer i's utility maximization behavior is represented in the following first-order condition

$$\begin{aligned}\mathrm{MRS}_i(x_i, y) = \frac{\partial u_i^\gamma(x_i, y)/\partial y}{\partial u_i^\gamma(x_i, y)/\partial x_i} = q_i\\ \iff x_i + \alpha_i - y = q_i(y + \varepsilon),\end{aligned}$$

together with the budget constraint $x_i+q_i y = w_i + \theta_i\pi$. On the other hand, the profit maximization bahavior gives us $p = C'(y) = 8y$, and $\pi = 4y^2$. Summing the first-order conditions over two consumers, I obtain the *Lindahl-Samuelson Condition*:

$$\mathrm{MRS}_1(x_1, y) + \mathrm{MRS}_2(x_2, y) = C'(y).$$

This can be written as

$$12y^2 + 2(4\varepsilon + 1)y - (\bar{w} + \alpha_1 + \alpha_2) = 0.$$

Notice that the level of the public good y determined by the above equation does not depend on the distribution of the private good between two consumers. For more details about independence of allocative efficiency from distribution of the private good, see Bergstrom and Cornes (1983).

In what follows, put $\varepsilon = 1$, $e_1 = (w_1, \alpha_1) = (3, 1)$, $e_2 = (w_2, \alpha_2) = (2, 2)$, and $(\theta_1, \theta_2) = (\frac{1}{2}, \frac{1}{2})$. Then, $y = \frac{1}{2}$. The corresponding price and the profit of the producer are $p = 4$ and $\pi = 1$. It is not difficult to see that the final distribution of the private good is $(x_1, x_2) = (\frac{5}{2}, \frac{3}{2})$, and the personalized prices are $(q_1, q_2) = (2, 2)$. The sum of consumers' personalized prices is equal to the producer's price, that is, $q_1 + q_2 = 2 + 2 = 4 = p$, indeed. Furthermore, the allocation is balanced. In other words, $x_1 + x_2 + C(y) = \frac{5}{2} + \frac{3}{2} + 4 \times (\frac{1}{2})^2 = 5 = w_1 + w_2$. □

3 Resource Allocation Processes

An ordered pair (ν, h) of a relation ν of E^γ into some set and a function h of range ν into $\mathbb{R}^I \times \mathbb{R}^K$ is called a *resource allocation process for* E^γ if for every $e \in E^\gamma$, $h(m) \in \mathcal{Z}(e)$ for every $m \in \nu(e)$. In other words, $h(\nu(e)) \subseteq \mathcal{Z}(e)$ for every element e of E^γ. For each allocation process (ν, h) for E^γ, ν is called the *equilibrium message correspondence*, h the *outcome function*, and $\nu(E^\gamma)$ the *message space*.

The present paper proposes the following resource allocation process which has desirable properties mentioned below. For each consumer i, define $M_i = \mathbb{R}^K_{++} \cup \{-1_K\}$, where 1_K is the unit vector in $\mathbb{R}^K$. Define

$M_J = \mathbb{R}^K_+$. Define $M = \prod_{i\in I} M_i \times M_J$. A generic element is represented as $m = ((m_i)_{i\in I}, m_J)$, where $m_i \in M_i$ for every consumer i and $m_J \in M_J$. Let m be an element of M. For each consumer i, define $q_{ik}(m) = m_i^k$ for each element k of K and $q_i(m) = (q_{ik}(m))_{k\in K}$. Finally, define $p(m) = \sum_{i\in I} q_i(m)$.

Let e an element of E^γ. Let m be any element of M. For each consumer i, define

$$\varphi_i(m, e_i) = \begin{cases} -1_K & \text{if } \zeta_i(m) \leqslant 0 \\ \dfrac{\nabla_y u_i^\gamma(\zeta_i(m), m_J)}{\gamma(m_J)} & \text{otherwise,} \end{cases}$$

where $\zeta_i(m) = w_i + \theta_i(p(m)\cdot m_J - C(m_J)) - q_i(m)\cdot m_J$, and $\nabla_y u_i^\gamma(x_i, y)$ is the gradient of $u_i^\gamma(x_i, y)$ with respect to the level of public goods y. For each element k of K, define

$$\varphi_J^k(m) = \begin{cases} 0 & \text{if } m_J^k = 0 \text{ and } \sum_{i\in I} m_i^k < \frac{\partial C(m_J)}{\partial y_k} \\ \sum_{i\in I} m_i^k - \frac{\partial C(m_J)}{\partial y_k} & \text{otherwise.} \end{cases}$$

Define $\varphi_J(m) = (\varphi_J^k(m))_{k\in K}$.

For each consumer i, define his equilibrium message correspondence by $\mu_i(e_i) = \{m \in M \mid m_i = \varphi_i(m, e_i)\}$. Define the equilibrium message correspondence of the government by $\mu_J(e_J) = \{m \in M \mid \varphi_J(m) = 0_K\}$. Define $\mu(e) = (\bigcap_{i\in I} \mu_i(e_i)) \cap \mu_J(e_J)$.[4] The equilibrium message correspondence μ represents the behavior of consumers and the government.

The intuition behind the constructed resource allocation process (μ, g) is the following. Consider a message $m = ((m_i)_{i\in I}, m_J) \in \mu(e)$ with $\varphi_i(m, e_i) \neq -1_K$ and $m_J \neq 0_K$. The government proposes an amount of public goods denoted by m_J. The amount of the private good assigned to consumer i is given by $\zeta_i(m) = w_i + \theta_i(p(m)\cdot m_j - C(m_J)) - q_i(m)\cdot m_J$, and then his announcement m_i is equal to the marginal rate of substitution evaluated at $(\zeta_i(m), m_J)$, that is,

$$\varphi_i(m, e_i) = \mathrm{MRS}_i(\zeta_i(m), m_J),$$

where

$$\mathrm{MRS}_i(x_i, y) = \frac{\nabla_y u_i^\gamma(x_i, y)}{\partial u_i^\gamma(x_i, y)/\partial x_i}.$$

Notice that the aggregate consumption of the private good proposed by the resource allocation process is given by $\sum_i \zeta_i(m) = \sum_{i\in I} w_i + p(m)\cdot m_J + p(m)\cdot C(m_J) - p(m)\cdot m_J = \bar{w} - C(m_J)$. Using this balancedness condition, the sum of the marginal rates of substitution based on the announcement m can be written as

$$\begin{aligned} &\textstyle\sum_{i\in I} \varphi_i(m, e_i) \\ &= \textstyle\sum_{i\in I} \mathrm{MRS}_i(\zeta_i(m), m_J) \\ &= \textstyle\sum_{i\in I} \dfrac{\nabla\gamma(m_J)\zeta_i(m) + \nabla v_i(m_J)}{\gamma(m_J)} \\ &= \gamma(m_J)^{-1}\left[\nabla\gamma(m_J)(\bar{w} - C(m_J)) \right. \\ &\qquad \left. + \textstyle\sum_{i\in I} \nabla v_i(m_J)\right]. \end{aligned}$$

Rearranging this expression to get

$$\begin{aligned} &\nabla\gamma(m_J)(\bar{w} - C(m_J)) \\ &= \gamma(m_J)\textstyle\sum_{i\in I} \varphi_i(m, e_i) - \sum_{i\in I} \nabla v_i(m_J). \end{aligned}$$

On the other hand, when the preferences of each consumer i are represented by the form

[4] Koyama (1989) considers a resource allocation process for an economy with increasing returns to scale in which there are a representative consumer and several producers. Because the marginal cost pricing leads losses when there are increasing returns to scale, the central agent imposes lump-sum transfers on the representative consumer in order to cover deficits. He analyzes the relationship between the stability and efficiency of a resource allocation process.

of $u_i^\gamma(x_i, y) = \gamma(y)x_i + v_i(y)$, using the balancedness condition $\sum_{i\in I} x_i + C(y) = \bar{w}$, the *aggregate* utility function in terms of the amount of public goods can be written as

$$U(y) = \gamma(y)(\bar{w} - C(y)) + \textstyle\sum_{i\in I} v_i(y).$$

Differentiating the aggregate utility function and evaluating it at m_J to obtain the following expression:

$$\begin{aligned}
&\nabla U(m_J)\\
&= \nabla\gamma(m_J)(\bar{w} - C(m_J))\\
&\quad -\gamma(m_J)\nabla C(m_J) + \textstyle\sum_{i\in I} \nabla v_i(m_J)\\
&= \gamma(m_J)\textstyle\sum_{i\in I} \varphi_i(m, e_i) - \sum_{i\in I} \nabla v_i(m_J)\\
&\quad -\gamma(m_J)\nabla C(m_J) + \textstyle\sum_{i\in I} \nabla v_i(m_J)\\
&= \gamma(m_J)\left[\textstyle\sum_{i\in I} \varphi_i(m, e_i) - \nabla C(m_J)\right].
\end{aligned}$$

Since $\gamma(m_J) > 0$, it follows from $m_J \neq 0_K$ that

$$\begin{aligned}
&\nabla U(m_J) = 0_K\\
&\iff \textstyle\sum_{i\in I} \varphi_i(m, e_i) = \nabla C(m_J)\\
&\iff \textstyle\sum_{i\in I} \mathrm{MRS}_i(\zeta_i(m), m_J) = \nabla C(m_J)\\
&\iff m \in \mu_J(e_J).
\end{aligned}$$

Therefore, the equilibrium message correspondence of the government reflects the Lindahl-Samuelson condition calculated in a decentralized way, and the problem to find the amount of public goods that maximizes the aggregate utility function.

It remains to define an outcome function. Let $e \in E^\gamma$. For each $m \in \mu(e)$, define

$$g^y(m) = \begin{cases} m_J \text{ if } p(m) = \nabla C(m_J) \\ 0_K \qquad\qquad \text{otherwise,} \end{cases}$$

and

$$\begin{aligned}
&g_i^x(m)\\
&= w_i + \theta_i(p(m)\cdot g^y(m) - C(g^y(m)))\\
&\quad -q_i(m)\cdot g^y(m)
\end{aligned}$$

for each consumer i. Define $g(m) = ((g_i^x(m))_{i\in I}, g^y(m))$. Notice also that if $p(m) \neq \nabla C(m_J)$, then $(g_i^x(m), g^y(m)) = (w_i, 0)$ for every consumer i, that is, it is autarkic.

I now verify that the ordered pair proposed above is well-defined in the sense that the outcome function always assign a feasible allocation for each message profile.

Proposition 1. *The ordered pair (μ, g) is a resource allocation process on E^γ.*

Proof. Let $m = ((m_i)_{i\in I}, m_J) \in \mu(e)$. Suppose first that $p(m) = \nabla C(m_J)$. Then, $g^y(m) = m_J$. Observe that

$$\begin{aligned}
&\textstyle\sum_{i\in I} g_i^x(m) + C(g^y(m))\\
&= \textstyle\sum_{i\in I} w_i + p(m)\cdot g^y(m) - C(g^y(m))\\
&\quad - \textstyle\sum_{i\in I} q_i(m)\cdot g^y(m) + C(g^y(m))\\
&= \textstyle\sum_{i\in I} w_i + p(m)\cdot g^y(m)\\
&\quad - \textstyle\sum_{k\in K}\left(\sum_{i\in I} q_{ik}(m) m_J^k\right)\\
&= \textstyle\sum_{i\in I} w_i + p(m)\cdot g^y(m)\\
&\quad - \textstyle\sum_{k\in K} p_k(m) m_J^k\\
&= \textstyle\sum_{i\in I} w_i + p(m)\cdot g^y(m) - p(m)\cdot g^y(m)\\
&= \textstyle\sum_{i\in I} w_i.
\end{aligned}$$

On the other hand, if $p(m) \neq \nabla C(m_J)$, then $g^y(m) = 0_K$. Since $C(0_K) = 0_K$, it follows that $\sum_{i\in I} g_i^x(m) + C(g^y(m)) = \sum_{i\in I} w_i + 0 = \sum_{i\in I} w_i$. By the definition of the outcome function g, the individual feasibility is obvious. Since m was arbitrary, it follows that $g(\mu(e)) \subseteq \mathcal{Z}(e)$. □

Some desirable properties of resource allocation processes are the following. A resource allocation process (ν, h) is said to *be non-wasteful on* E^γ if $h(\nu(e)) \subseteq PT(e)$ for every element e of E. The non-wastefulness describes the performance of a resource allocation process. The next property captures the idea that knowledge is dispersed. A resource allocation process (ν, h) is said to *be informationally decentralized on* E^{γ} if there is a correspondence δ on $I \cup J$ such that $\nu(e) = \left(\bigcap_{i \in I} \delta_i(e_i)\right) \cap \delta_J(e_J)$ for every element e of E^{γ}. By construction of (μ, g), the following assertion is obvious.

Proposition 2. *The resource allocation process* (μ, g) *is informationally decentralized on* E^{γ}.

The intuition behind my formulation is that the government proposes a public goods allocation, and consumers respond with marginal rates of substitution, and the government revises the amount of public goods on the basis of announced marginal rate of substitutions. The correspondence μ from the environment E^{γ} into the message space M represents a communication among consumers and the government. I may consider a resource allocation process in which communication takes place by an iterative exchange of messages, based on previously received messages. In the literature on informational efficiency, however, a *one-step* communication process is considered. Each agent sends a proposed joint messages that are acceptable to him, given his economic environmental component only. A new message will be proposed until every agent accepts it. This interpretation is referred to as the *verification scenario*. In this sense, equilibrium messages indicate how the decisions of agents have been coordinated. These equilibrium messages are then mapped by the outcome function into the allocations. If all the outcomes are the ones prescribed by a choice rule, then it is said that the resource allocation process *fully realizes* that choice rule. The problem of designing (μ, g) is described in Figure 1. More details about message exchange processes can be found in Hurwicz and Reiter (2008).

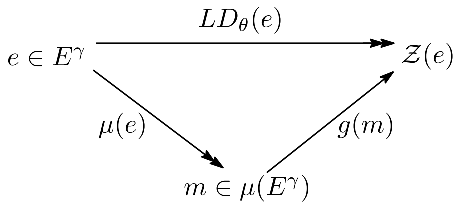

Figure 1: Realization of $LD_\theta(e)$

4 Main Results

This section is devoted to state the equivalence of the Lindahl choice rule and the resource allocation process (μ, g). The following lemmas are preliminary results used to establish the main result.

Lemma 1. *For every* $m \in \mu(E^{\gamma})$, *each consumer* i *is assigned a positive amount of the private good,* $g_i^x(m) > 0$. *Furthermore, personalized prices are positive,* $q_i(m) \gg 0_K$, *for every consumer* i.

Proof. Consider any $m \in \mu(E^{\gamma})$. Suppose, by way of contradiction, that $g_i^x(m) = 0$ for some consumer i. If $g^y(m) = 0_K$, then it must be the case that $g_i^x(m) = w_i > 0$, a contradiction. Therefore, $g^y(m) \neq 0_K$, and hence $p(m) = \nabla C(m_J)$ and $g^y(m) = m_J$ by the definition of the outcome function. Recall that the marginal cost pricing scheme is sufficient for profit maximization by the strict convexity of the cost function. Since zero-profit is always possible, it follows that the maximized profit must be non-negative, that is, $p(m) \cdot g^y(m) - C(g^y(m)) \geqslant 0$. Since $g^y(m) = m_J$, it follows that $g_i^x(m) = w_i + \theta_i(p(m) \cdot g^y(m) - C(g^y(m))) - q_i(m) \cdot g^y(m)) = w_i + \theta_i(p(m) \cdot m_J - C(m_J)) -$

$q_i(m) \cdot m_J$. This yields that $\zeta_i(m) = 0$. Furthermore, the expenditure on public goods is positive, $q_i(m) \cdot g^y(m) > 0$, and hence $q_i(m) \neq 0_K$. However, $\varphi_i(m, e_i) = -1_K$ holds whenever $\zeta_i(m) = 0$. This, together with $m \in \mu_i(e_i)$, implies that $q_i(m) = m_i = -1_K \ll 0_K$. This is a contradiction. □

The above lemma will not rule out the possibility of no production of public goods in general. The next lemma examines the implication of the following regularity condition on the cost function.

Assumption 3. $(\nabla C)^{-1}(0_K) = \{0_K\}$.

One may see that, for instance, the cost function $C(y) = \sum_{k \in K} c_k y_k^2$ for some $c = (c_k)_{k \in K} \gg 0_K$ satisfies the above assumption.

Lemma 2. *Under Assumption 3,* $g^y(m) \neq 0_K$ *for every* $m \in \mu(E^\gamma)$ *such that* $p(m) = \nabla C(m_J)$.

Proof. Consider any $m \in \mu(E^\gamma)$. Suppose, by way of contradiction, that $g^y(m) = 0_K$. The previous lemma ensures that $p(m) = \sum_{i \in I} q_i(m) \gg 0_K$. Applying the marginal cost pricing condition, $p(m) = \nabla C(m_J)$, to obtain $g^y(m) = m_J$. By the hypothesis, $m_J = 0_K$, and thus $p(m) = \nabla C(0_K)$. Assumption 3 implies that $p(m) = 0_K$ must hold, which is a contradiction. □

Lemma 3. *Under Assumption 3,* $g^y(m) \neq 0_K$ *if and only if* $p(m) = \nabla C(m_J)$ *for every* $m \in \mu(E^\gamma)$.

Proof. I have shown in Lemma 2 that $g^y(m) \neq 0_K$ must hold whenever the marginal cost pricing scheme prevails. The reverse implication is straightforward by the definition of the outcome function. □

Corollary 1. *Under Assumption 3, if there is a single public good, that is,* $\#K = 1$, *then the resource allocation process* (μ, g) *always achieves an interior feasible allocation if and only if the marginal cost pricing condition holds for every* $m \in \mu(E^\gamma)$.

Proof. Immediate from Lemmas 1 and 3. □

According to the construction of the outcome function, the autarkic situation, $g(m) = ((w_i)_{i \in I}, 0_K)$, is possible for some $m \in M$. Lemma 3 states that this will not happen for any equilibrium message profile $m \in \mu(E^\gamma)$ subject to the marginal cost pricing condition if the cost function satisfies the regularity condition. An intermediate observation is that all of equilibrium outcomes are not on the boundary of the set of feasible allocations on E^γ if there is only one public good in the same circumstances. I put the following assumption to simplify the argument.

Assumption 4. For every $m \in \mu(E^\gamma)$, $g^y(m) \gg 0_K$ if and only if $p(m) = \nabla C(m_J)$.

The main result of the present paper is the following. It is shown that the above informationally decentralized process *fully realizes* the Lindahl choice rule.

Theorem 1 (Realization). *Under Assumptions 1 to 4,* $LD_\theta(e) = g(\mu(e))$ *for every element* e *of* E^γ.

Proof. See the Appendix. □

Corollary 2. *Under Assumptions 1 to 4, the resource allocation process* (μ, g) *is nonwasteful on* E^γ.

Proof. Since any Lindahl allocation is Pareto optimal on the environment E^γ, the assertion is immediate from Theorem 1. □

5 Concluding Remarks

The present paper analyzes a well-behaved resource allocation process whose outcomes coincide with Lindahl allocations when utility functions are of the special form, proposed by Bergstrom and Cornes (1983). The resource allocation process reflects the idea of informational decentralization. During a process of information exchange among the consumers and the government, each agent responds based only on his own characteristics. The formulation of the resource allocation process crucially depends on the class of utility functions so as to admit construction of a representative consumer's utility function appeared in the equilibrium message correspondence of the government.

Sincere behavior for consumers are assumed throughout the paper. In other words, each consumer's response is based on his true preferences and initial endowment. However, consumers may benefit by misrepresenting their characteristics strategically. There is a well-known argument that consumers have no incentive to reveal their *true* preferences for financing public goods. In contrast to my approach, Li, et al. (1995) consider the Nash-implementability of the Lindahl choice rule in the presence of decreasing returns to scale technologies. The mechanism or the game form proposed by Li, et al. (1995) does not require information about consumers' preferences, however, there must be at least three consumers due to the construction of their mechanism. On the other hand, the mechanism in the present paper works even when there are only two consumers, but the mechanism designer needs to know information about consumers' preferences.

Appendix: Proof of Theorem 1

Proof. Consider any $e \in E^\gamma$. The proof consists of two steps.

Step 1. $g(\mu(e)) \subseteq LD_\theta(e)$.

Proof of Step 1. Let $((x_i^*)_{i\in I}, y^*)$ be any element of $g(\mu(e))$. Then, $((x_i^*)_{i\in I}, y^*) = g(m)$ for some element m of $\mu(e)$, where $m = ((m_i)_{i\in I}, m_J)$. Then, $x_i^* = g_i^x(m) > 0$ for every consumer i, by Lemma 1, and $y^* = g^y(m)) \gg 0_K$ by Assumption 4. By Lemma 3, the marginal cost pricing scheme prevails, which is sufficient for profit maximization by the strict convexity of the cost function, that is, $y^* \in \text{argmax}\ [p(m)\cdot y - C(y) \mid y \in \mathbb{R}_+^K]$. It remains to analyze consumers' utility maximization behavior.

Consider any consumer i. Since $m \in \mu_i(e_i)$, it follows that either $\varphi_i(m, e_i) = m_i \gg 0_K$ or $\varphi_i(m, e_i) = -1_K$. Since $g^y(m) \neq 0_K$ is equivalent to the marginal cost pricing condition, $p(m) = \nabla C(m_J)$, by Lemma 3, it follows that $y^* = g^y(m) = m_J$, and hence $x_i^* = g_i^x(m) = w_i + \theta_i(p(m)\cdot y^* - C(y^*)) - q_i(m)\cdot y^*$. I conclude that $\zeta_i(m) = x_i^* > 0$. Thus, I have $m_i = \nabla_y u_i^\gamma(\zeta_i(m), m_J)/\gamma(m_J) = \nabla_y u_i^\gamma(x_i^*, y^*)/\gamma(y^*)$, or equivalently,

$$m_i = \frac{\nabla\gamma(y^*)}{\gamma(y^*)}x_i^* + \frac{\nabla v_i(y^*)}{\gamma(y^*)}.$$

For each element (x_i, y) of $\mathbb{R}_+^H$, form the Lagrangian $\mathcal{L}_i(x_i, y) = u_i^\gamma(x_i, y) + \lambda h_i(x_i, y)$, where $\lambda = \gamma(y^*) > 0$ and $h_i(x_i, y) = w_i + \theta_i(p(m)\cdot y^* - C(y^*)) - x_i - q_i(m)\cdot y$. It is shown that the sufficient conditions for a strict local maximum will be met.

Claim 1. *(x_i^*, y^*) is a strict local maximum of $u_i^\gamma(x_i, y)$ subject to the budget constraint with equality.*

Proof of Claim 1. Firstly, I shall show that (x_i^*, y^*) is a stationary point of the Lagrangian $\mathcal{L}_i$. I see that $\frac{\partial \mathcal{L}}{\partial x_i}(x_i^*, y^*) = \gamma(y^*) - \lambda = \gamma(y^*) - \gamma(y^*) = 0$. Also, $\nabla_y \mathcal{L}(x_i^*, y^*) = \nabla_y u_i^\gamma(x_i^*, y^*) - \lambda q_i(m) = \nabla\gamma(y^*)x_i^* + \nabla v_i(y^*) - \lambda q_i(m)$. Use $m_i = \frac{\nabla\gamma(y^*)x_i^*}{\gamma(y^*)} + \frac{\nabla v_i(y^*)}{\gamma(y^*)}$ to obtain $\lambda q_i(m) = \gamma(y^*)m_i = \nabla\gamma(y^*)x_i^* + \nabla v_i(y^*)$, or equivalently, $\nabla_y \mathcal{L}(x_i^*, y^*) = 0_K$. Therefore, (x_i^*, y) satisfies the first-order conditions indeed, that is, $\nabla \mathcal{L}(x_i^*, y^*) = 0_H$. Consider any element z of $\mathbb{R}^H \setminus \{0_H\}$ such that $\nabla h(x_i^*, y^*) \cdot z = 0$. It suffices to show that the Hessian $D^2\mathcal{L}(x_i^*, y^*)$ is negative definite subject to $\nabla h(x_i^*, y^*) \cdot z = 0$. One may see that

$$\begin{aligned} &D^2\mathcal{L}(x_i^*, y^*) \\ &= \begin{bmatrix} 0, & \nabla\gamma(y^*) \\ [\nabla\gamma(y^*)]^T, & D^2\gamma(y^*)x_i^* + D^2 v_i(y^*) \end{bmatrix} \\ &= D^2 u_i^\gamma(x_i^*, y^*). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\nabla u_i^\gamma(x_i^*, y^*) \cdot z \\ &= \Big[\gamma(y^*),\, \nabla\gamma(y^*)x_i^* + \nabla v_i(y^*)\Big] \cdot z \\ &= \Big[\lambda,\, \lambda q_i(m)\Big] \cdot z \\ &= -\lambda \nabla h(x_i^*, y^*) \cdot z = 0. \end{aligned}$$

By Assumption 2, I conclude that $zD^2\mathcal{L}(x_i^*, y^*) \cdot z = zD^2 u_i^\gamma(x_i^*, y^*) \cdot z < 0$ subject to $\nabla h(x_i^*, y^*) \cdot z = 0$. This establishes the claim.

By Claim 1, there exists a positive real number δ such that $u_i^\gamma(x_i^*, y^*) > u_i^\gamma(x_i, y)$ for every $(x_i, y) \in \mathbb{R}_+^H$ such that $\| (x_i^*, y^*) - (x_i, y) \| < \delta$ and $x_i + q_i(m) \cdot y = w_i + \theta(p(m) \cdot y^* - C(y^*))$. Next, I shall show that (x_i^*, y) is a global maximum subject to the budget constraint with inequality.

Claim 2. *$u_i^\gamma(x_i^*, y^*) \geqslant u_i^\gamma(x_i, y)$ for every element $(x_i, y) \in \gamma_i(q_i(m), p(m) \cdot y^* - C(y^*), e_i)$.*

Proof of Claim 2. Consider any element (x_i, y) of $\mathbb{R}_+^H$ such that $x_i + q_i(m) \cdot y \leqslant w_i + \theta_i(p(m) \cdot y^* - C(y^*))$. If $(x_i, y) \in \partial\mathbb{R}_+^H$, then it follows from Assumption 1 that the interior point (x_i^*, y^*) is strictly preferred to (x_i, y). It remains to show that $u_i^\gamma(x_i^*, y^*) \geqslant u_i^\gamma(x_i, y)$ if $(x_i, y) \in \mathbb{R}_{++}^H$. Suppose, by way of contradiction, that there is some element $(x_i, y) \in \mathbb{R}_{++}^H$ such that $u_i^\gamma(x_i, y) > u_i^\gamma(x_i^*, y^*)$ and $x_i + q_i(m) \cdot y \leqslant w_i + \theta_i(p(m) \cdot y^* - C(y^*))$. By letting $(x_i(t), y(t)) = (1-t)(x_i, y) + t(x_i^*, y^*)$, one may see that $\| (x_i(t), y(t)) - (x_i^*, y^*) \| < \delta$ and $(x_i(t), y(t)) \gg 0_H$ for some sufficiently large element t of $]0,1[$. Besides, by the strict monotonicity and strict quasi-concavity of preferences, there exists some element $(\hat{x}_i, \hat{y}) \gg 0_H$ in the neighborhood of (x_i^*, y^*) with radius δ such that $u_i^\gamma(\hat{x}_i, \hat{y}) > u_i^\gamma(x_i^*, y^*)$, located above the budget line. Finally, I can take a linear combination of $(x_i(t), y(t))$ and $(\hat{x}_i, \hat{y})$ on the budget line, denoted by $(\tilde{x}_i, \tilde{y})$, such that $u_i^\gamma(\tilde{x}_i, \tilde{y}) > u_i^\gamma(x_i^*, y^*)$. However, $(\tilde{x}_i, \tilde{y})$ also belongs to the neighborhood of (x_i^*, y^*) with radius δ, which contradicts Claim 1 (cf. Figure 2).

Claims 1 to 2 establish the step.

Step 2. $LD_\theta(e) \subseteq g(\mu(e))$.

Proof of Step 2. Let $((x_i^*)_{i\in I}, y^*)$ be any element of $LD_\theta(e)$. There exists an element $q = (q_i)_{i\in I}$ of $(\mathbb{R}_{++}^K)^I$ with $\sum_{i\in I} q_i = p$ and $p \cdot y^* - C(y^*) = \max\{p \cdot y - C(y) \mid y \in \mathbb{R}_+^K\}$. Consider any consumer i. I shall show first that $(x_i^*, y^*) \gg 0_H$. Suppose, by way of contradiction, that $(x_i^*, y^*) \in \partial\mathbb{R}_+^H$. Since zero-profit is always possible from no production

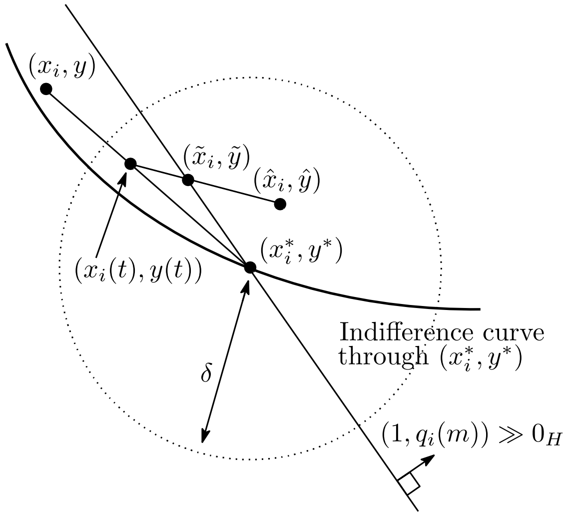

Figure 2: Construction of $(\tilde{x}_i, \tilde{y})$ in Claim 2

of public goods, it follows that the maximized profit must be non-negative, that is, $p \cdot y^* - C(y^*) \geqslant 0$. This, together with $w_i > 0$, implies the existence of some element $(x_i, y) \gg 0_H$ satisfying the budget constraint of consumer i. Then, $u_i^\gamma(x_i, y) > u_i^\gamma(x_i^*, y^*)$ under Assumption 1, which contradicts the fact that $(x_i^*, y^*) \in D_i(q_i, p \cdot y^* - C(y^*), e_i)$. Therefore, $(x_i^*, y^*) \gg 0_H$.

It remains to show that $((x_i^*)_{i \in I}, y^*) = g(m)$ for some element m of $\mu(e)$. Define $m = ((m_i)_{i \in I}, m_J)$ by $m_i = \nabla_y u_i^\gamma(x_i^*, y^*)/\gamma(y^*)$ for each consumer i, and $m_J = y^*$. Clearly, $m \in M$.

Claim 1. *$m \in \mu_i(e_i)$ for every consumer i.*

Proof of Claim 1. Consider any consumer i. For each element (x_i, y) of $\mathbb{R}_+^H$, define $h(x_i, y) = w_i + \theta_i(p \cdot y^* - C(y^*)) - x_i - q_i \cdot y$. Since the usual constraint qualification is satisfied, there exists a real number λ such that $\nabla u_i^\gamma(x_i^*, y^*) + \lambda \nabla h(x_i^*, y^*) = 0_H$, or equivalently, $\gamma(y^*) = \lambda$ and $\nabla \gamma(y^*) x_i^* + \nabla v_i(y^*) = \lambda q_i$. This yields that $q_i = \frac{\nabla \gamma(y^*)}{\gamma(y^*)} x_i^* + \frac{\nabla v_i(y^*)}{\gamma(y^*)} = \nabla_y u_i^\gamma(x_i^*, y^*)/\gamma(y^*) = m_i$, and hence $q_i(m) = q_i$. Also, $p(m) = \sum_{i \in I} q_i(m) = \sum_{i \in I} q_i = p$. Note that $\zeta_i(m) = w_i + \theta_i(p \cdot y^* - C(y^*)) - q_i \cdot y^* = x_i^* > 0$, which implies that $\varphi_i(m, e_i) = \nabla_y u_i^\gamma(\zeta_i(m), m_J)/\gamma(m_J) = \nabla_y u_i^\gamma(x_i^*, y^*)/\gamma(y^*) = m_i$. Therefore, $m \in \mu_i(e_i)$.

Claim 2. $m \in \mu_J(e_J)$.

Proof of Claim 2. Notice that since y^* is an interior profit maximizing level of public goods, it follows that the marginal cost pricing scheme, $p = \nabla C(y^*)$, is necessary. Since $m_J = y^* \gg 0_K$ and $m_i = q_i$ for every consumer i, it follows that for every $k \in K$, $\varphi_J^k(m) = \sum_{i \in I} m_i^k - \frac{\partial C(m_J)}{\partial y_k} = \sum_{i \in I} q_{ik} - \frac{\partial C(y^*)}{\partial y_k} = p_k - \frac{\partial C(y^*)}{\partial y_k} = 0$, where the equality holds due to the marginal cost pricing condition, $p = \nabla C(y^*)$. Therefore, $\varphi_J(m) = 0_K$, and hence $m \in \mu_J(e_J)$.

Claims 1 and 2 establish Step 2. □

References

[1] Theodore C. Bergstrom and Richard C. Cornes. Independence of allocative efficiency from distribution in the theory of public goods. *Econometrica*, 51(6):1753–65, November 1983.

[2] Duncan K. Foley. Resource allocation and the public sector. *Yale Economic Essays*, 7(1):45–98, Spring 1967.

[3] Leonid Hurwicz. A perspective. In Leonid Hurwicz, David Schmeidler, and Hugo Sonnenschein, editors, *Social Goals and Social Organization: Essays in Memory of Elisha Pazner*, pages 1–16. Cambridge University Press, December 1986.

[4] Leonid Hurwicz and Stanley Reiter. *Designing Economic Mechanisms*. Cambridge University Press, December 2008.

[5] Kouichi Koyama. Mechanism Design for an Economy with Increasing Returns to Scale. Ph.D. Dissertation, University of Minnesota, 1989.

[6] Qi Li, Shinsuke Nakamura, and Guoqiang Tian. Nash-implementation of the lindahl correspondence with decreasing returns to scale technologies. *International Economic Review*, 36(1):37–52, February 1995.

[7] Jean-Claude Milleron. Theory of value with public goods: A survey article. *Journal of Economic Theory*, 5(3):419–477, December 1972.