Nash-Implementation by Demand Revelation Mechanisms in Two-Sided Matching Models

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Abstract

I examine demand revelation mechanisms in many-to-one matching problems. Each agent announces an assignment or a set of assignments without indicating rankings. I show that any subcorrespondence of the Pareto choice rule is not Nash-implementable by any demand revelation mechanism. A stronger notion of Maskin monotonicity plays a crucial role for this impossibility result.

Keywords: Implementation · Mechanism Design · Demand Revelation · Maskin Monotonicity

1 Introduction

Two-sided matching problems consist of two finite disjoint sets of agents. Agents on one side need to be matched with agents on the other side. In the classical formulation of these problems, a matching is described only by the identities of the partners, that is, which pairs are matched. Hatfield and Milgrom (2005) develop a more general class of many-to-one matching problems in which matches are characterized by contracts as well as by who is matched. A contract specifies terms and conditions regarding the match, for instance, stipend, working hours, wage, etc.

A well-known practical matching procedure is the deferred acceptance algorithm proposed by Gale and Shapley (1962), and generalized by Hatfield and Milgrom (2005) to apply to matching with contracts. For example, in the context of the labor market for medical interns without contracts, the doctor-proposing deferred acceptance algorithm finds a matching through the following steps. At the first step, every doctor applies to his favorite acceptable hospital (if there is no such hospital then he remains unmatched). Each hospital tentatively assigns its seats to the favorite subset of doctors from the pool of applicants; the remaining doctors are rejected. At the kth step, those applicants who were rejected at step $k-1$ apply to their next best acceptable hospitals. Each hospital tentatively assigns its seats to the most favorable subset of doctors from the pool of new applicants and those tentatively assigned to that hospital in the previous step; the remaining doctors are rejected. The algorithm terminates when every doctor is either held tentatively by some hospital or has been rejected by every hospital that is acceptable to him. In this kind of algorithm, participants in the market are asked to submit *rankings or preferences* used for determining the final matching. However, when the preferences of both sides are private information, it is known that reporting the true preference profile is neither a dominant strategy equilibrium nor a Nash equilibrium of the *preference revelation game* induced by the deferred acceptance algorithm (see Roth 1982, Theorem 3).

On the other hand, I consider a class of mechanisms refereed to as *demand revelation mechanisms*. Agents submit lists of preferred matchings without indicating priorities in demand revelation mechanisms. I call a mechanism in which proposals are restricted to single matching a demand revelation mechanism with final offers. I call a mechanism in which proposals are not restricted to single matching a demand revelation mechanism with general offers. It can be interpreted as the situation in which agents submit the most preferred matchings with equal priority. I compare the set of non-cooperative outcomes with a set of Pareto efficient allocations with respect to the true pnreferences which are private information. It is shown that any subset of Pareto efficient allocations can not be implementable in Nash equilibrium of demand revelation games. As a result, I obtain the impossibility result of Nash-implementability of the core correspondence.

2 Environments

The set of finite agents is denoted by I, where $\#I \geqslant 3$. There is a partition of I, denoted by (D, H). Both D and H are non-empty. The two disjoint sets are interpreted as the set D of doctors and the set H of hospitals. Denote by d and h generic elements of D and H, respectively. Agents on one side of the market need to be matched with agents on the other side.

Denote by X a non-empty finite set. An element of X is called a *contract*. Each non-empty subset X' of X is called an *allocation*, which is a collection of contracts. Each contract is *bilateral* in the sense that each contract is associated with a pair of agents in $D \times H$. For each contract $x \in X$, the ordered pair (x_D, x_H) represents a pair of agents $x_D \in D$ and $x_H \in H$, associated through the contract x. Abusing notation, $x_I \in \{x_D, x_H\}$. Agent i is involved in an allocation X' if there exists some contract $x \in X'$ such that $x_I = i$. Otherwise, agent i is outside X'. Each agent may stay unmatched. I refer to the situation where an agent is unmatched as the *null contract*. The null contract matches an agent to himself. Denote by $\emptyset$ the null contract. The null contract can be assigned to several agents. The set $A = 2^X$ will be referred to as the allocation space.

The distinction between agents in D and H comes from the difference in definition of consumption sets. Agents have private information about their preferences over possible matches. Roughly speaking, agents in D have preferences over individual contracts, whereas agents in H have preferences over groups of contracts. For each agent $d \in D$, his consumption set is defined by

$$\mathcal{X}_d = \{\{x\} \subseteq X \mid x_D = d\} \cup \{\emptyset\}.$$

The fact that $\mathcal{X}_d$ consists of singletons means that each agent $d \in D$ can sign at most one contract. Each agent $d \in D$ has a preference relation R_d defined on $\mathcal{X}_d$, which is a linear order.[1] In contrast, each agent $h \in H$ has a more complicated consumption set. For each agent $h \in H$, his consumption set is de-

[1] A preference relation is a linear order if it is complete, transitive and antisymmetric.

fined by

$$\mathcal{X}_h = \{B_h \subseteq X \mid x_H = h = y_h \text{ and } [x_D = y_D \text{ implies } x = y] \text{ for every } x, y \in B_h\} \cup \{\emptyset\}.$$

Each agent h can sign only one contract with each agent involved in B_h. Each agent $h \in H$ has a preference relation R_h defined on $\mathcal{X}_h$, which is a linear order. Lastly, for each agent $i \in I$, denote the associated strict part of R_i by P_i and indifference by I_i.

I will denote the most preferred set of agent $i \in I$ under R_i by $C_{R_i}(X')$ for each allocation X'. Notice that for each agent $d \in D$, his chosen set $C_{R_d}(X')$ is either a singleton of the null contract or a singleton of a contract, whereas the chosen set $C_{R_h}(X')$ of agent $h \in H$ is a subset of the contracts associated with him within the allocation X'.

The literature on many-to-one matching problems introduces a certain restriction to preferences of agents in H.

Definition 1. For each agent $h \in H$, his preference relation R_h is *weakly substitutable*[2] if for every $B'_h \in \mathcal{X}_h$ and every $B''_h \subseteq X$ such that $B'_h \subseteq B''_h$ and $x_H = h$ for every $x \in B''_h$, $B'_h \cap C_{R_h}(B''_h) \subseteq C_{R_h}(B'_h)$.

Denote by $\mathcal{R} = \prod_{i \in I} \mathcal{R}_i$ the weakly substitutable preference domain.

For each allocation $X' \in A$ and each preference profile $R \in \mathcal{R}$, denote $R_D = (R_d)_d$ where d involved in X', and $R_H = (R_h)_h$ where h involved in X'. Both profiles R_D and R_H are constructed with respect to X', but I do not make this dependence explicit. Consider any preference profile $R \in \mathcal{R}$. For each allocation $X' \in A$, denote $C_{R_D}(X') = \bigcup\{C_{R_d}(X') \mid x_D = d \text{ for some } x \in X'\}$ and $C_{R_H}(X') = \bigcup\{C_{R_h}(X') \mid x_H = h \text{ for some } x \in X'\}$. Then, an allocation X' is *balanced for* R if $C_{R_D}(X') = C_{dR_H}(X') = X'$.

My analysis takes a *choice rule* or *performance standard* as a given. A choice rule φ is a correspondence of $\mathcal{R}$ into A that associates with each preference profile a set of balanced allocations regarded as desirable for the profile. In other words, each element $X' \in \varphi(R)$ is a socially desirable allocation under the preference profile R with respect to φ. A balanced allocation X' for R is said to *be Pareto efficient for* R if there is no balanced allocation X'' for R that Pareto dominates X': $C_{R_i}(X'')R_iC_{R_i}(X')$ for every agent i and $C_{R_i}(X'')P_iC_{R_i}(X')$ for some agent i. The relation φ_{PE} of $\mathcal{R}$ into A is referred to as the *Pareto choice rule*. I seek selections from the Pareto choice rule. Denote by $\mathcal{I} \subseteq 2^I \setminus \{\emptyset\}$ the set of admissible coalitions. A balanced allocation X' is said to *be in the core for* R if there is no balanced allocation X'' with respect to coalition $J \in \mathcal{I}$ that *weakly* blocks X' via coalition J under R, where J is the set of agents involved in X''.[3] Denote by $\varphi_C(R)$ the core for R. The relation

[2] A stronger notion of weak substitutability is introduced by Hatfield and Milgrom (2005). They request $B'_h \cap C_{R_h}(B''_h) \subseteq C_{R_h}(B'_h)$ whenever $B'_h \subseteq B''_h \subseteq A$. The inclusion $B'_h \subseteq B''_h$ in the allocation space rather than in his comsumption set does not necessarily imply that $\{x \in B'_h \mid x_H = h\} \subseteq \{x \in B''_h \mid x_H = h\}$.

[3] A balanced allocation X' is said to *be in the weak core for* R if there is no balanced allocation X'' with respect to coalition $J \in \mathcal{I}$ such that $C_{R_i}(X'')P_iC_{R_i}(X')$ for every $i \in J$, where J is the set of agents involved in X''. In other words, there is no balanced allocation with respect to coalition J that *strongly* blocks X' via coalition J under R. Denote by $\varphi_{WC}(R)$ the weak core for R. It is known that the two cores coincide in marriage problems, but not in the college admissions problems (see Roth and Sotomayor 1990, Proposition 5.36).

φ_C of $\mathcal{R}$ into A is referred to as the *core correspondence*. Notice that $\varphi_C(R) \subseteq \varphi_{PE}(R)$ for each $R \in \mathcal{R}$.

The reason of the imposition of weak substitutability is the following.

Remark 1. *The core correspondence φ_C is non-empty-valued if preferences are weakly substitutable.*[4]

3 Demand Revelation Games

The mechanism designer desires the outcomes described by a choice rule but does not know individual preferences which are private information of the agents. The task of the mechanism designer is to construct a set of rules, which is independent of private information, that achieves the prescribed socially desirable allocations as non-cooperative equilibrium outcomes. An ordered pair (M, g) is called a *mechanism* if g is a function of M into the allocation space A, and $M = \prod_{i \in I} M_i$, where M_i is a non-empty set for each agent i. The Cartesian product M is called the *message space*. Each element $m \in M$ is called a *message profile*. Given a preference relation R, the ordered pair (M, g) induces a game in normal form. A triplet (M, g, R) is called a *game* if (M, g) is a mechanism and $R \in \mathcal{R}$.

For each message profile $m = (m_{-i}, m_i)$, denote by $g_i(m_{-i}, m_i) = \{x \in g(m_{-i}, m_i) \mid x_I = i\}$ the set of contracts that agent i receives at the allocation $g(m_{-i}, m_i)$. Given a game (M, g, R), a message profile $m = (m_{-i}, m_i)$ is called an *equilibrium* or a *pure strategy Nash equilibrium for* (M, g, R) if for every agent i, $g_i(m_{-i}, m_i)\, R_i\, g_i(m_{-i}, \hat{m}_i)$ for every $\hat{m}_i \in M_i$, where the notation $(m_{-i}, \hat{m}_i)$ stands for the message profile obtained from the message profile (m_{-i}, m_i) by replacing its component m_i by $\hat{m}_i$. The set of equilibria for (M, g, R) is denoted by $\mathcal{N}_{(M,g)}(R)$. A mechanism (M, g) *Nash-implements* a choice rule φ if the equilibrium outcomes coincide with the set of $\varphi-$optimal outcomes for (M, g, R), that is, $g(\mathcal{N}_{(M,g)}(R)) = \varphi(R)$ for every $R \in \mathcal{R}$. The expression $g(\mathcal{N}_{(M,g)}(R))$ stands for the image of the set of equilibria; $g(\mathcal{N}_{(M,g)}(R)) = \bigcup\{g(m) \mid m \in \mathcal{N}_{(M,g)}(R)\}$.

The set of equilibrium messages can be decomposed as $\mathcal{N}_{(M,g)})(R) = \bigcap_{i \in I} \mathcal{N}^i_{(M,g)}(R_i)$ for every $R \in \mathcal{R}$, where $\mathcal{N}^i_{(M,g)}(R_i) = \{m \in M \mid g_i(m_i, m_{-i})\, R_i\, g_i(m_{-i}, \hat{m}_i)$ for every $\hat{m}_i \in M_i\}$ is the graph of agent i's best response correspondence at R_i. Notice that each correspondence $\mathcal{N}^i_{(M,g)}$ of $\mathcal{R}_i$ into M depends only on his own type R_i. In other words, the correspondence $\mathcal{N}_{(M,g)}$ of $\mathcal{R}$ into M is a coordinate correspondence.[5] The following figure depicts the notion of Nash-implementation.

$$\begin{tikzcd}
R \in \mathcal{R} \arrow[rr, "\varphi(R) = (g \circ \mathcal{N}_{(M,g)})(R)"] \arrow[dr, "\mathcal{N}_{(M,g)}(R)"'] & & X' \in A \\
& m \in M \arrow[ur, "g(m)"'] &
\end{tikzcd}$$

Figure 1: Nash-Implementation of φ by (M, g)

[4] The proof is available upon request.

[5] This is a privacy requirement on the equilibrium message correspondence (see Mount and Reiter 1974).

3.1 Mechanisms with Final Offers

In this subsection, I consider a class of mechanisms in which a message consists of a single contract. Each agent can submit at most one acceptable assignment, that is, I restrict my attention to the class of mechanisms, where $M_i = \mathcal{X}_i$ for each agent i. I consider the following class of outcome functions. For each message profile $m = (m_{-i}, m_i) \in M$, denote by $g_i(m_{-i}, m_i)$ the set of contracts that agent i must consume:

$$g_i(m_{-i}, m_i) = \begin{cases} m_i \text{ for } m_i \in \Gamma_i(m_{-i}) \\ \emptyset \quad \text{otherwise} \end{cases}$$

for some correspondence $\Gamma_i(m_{-i}) \subseteq \mathcal{X}_i \setminus \{\emptyset\}$. The outcome function is defined by

$$g(m_{-i}, m_i) = \bigcup_{g_i(m_{-i}, m_i) \neq \emptyset} g_i(m_{-i}, m_i).$$

The ordered pair (M, g) is called a *demand revelation mechanism with final offers*. In any demand revelation mechanism with final offers, the message space is defined by $M = \prod_{i \in I} \mathcal{X}_i$, and the attainable set of agent i is given by $\Gamma_i(m_{-i}) \cup \{\emptyset\}$. There is no particular restriction to the constraints $(\Gamma_{-i}(\cdot), \Gamma_i(\cdot))$.

The following theorem and corollary show that no matter how I construct constraint correspondences $(\Gamma_{-i}(\cdot), \Gamma_i(\cdot))$, it is impossible to achieve any set of Pareto efficient allocations and, in particular, the core.

Theorem 1. *Any subcorrespondence of the Pareto choice rule is not Nash-implementable by any demand revelation mechanism with final offers.*

Proof. See Appendix 1. □

Corollary 1. *The core correspondence φ_C is not Nash-implementable by any demand revelation mechanism with final offers.*

3.2 Mechanisms with General Offers

In the rest of the present note, I consider a more general message space consisting of multiple assignments. Each agent can submit a list of contracts without indicating priorities. The message space of each agent i is given by $M_i = \{m_i \mid m_i \subseteq \mathcal{X}_i \setminus \{\emptyset\} \text{ or } m_i = \emptyset\}$. I consider the following class of outcome functions. For each message profile $m = (m_{-i}, m_i) \in M$, denote by $g_i(m_{-i}, m_i)$ the set of contracts that agent i must consume:

$$g_i(m_{-i}, m_i) = \begin{cases} B_i \text{ for } B_i \in m_i \text{ and } B_i \in \Gamma_i(m_{-i}) \\ \emptyset \quad \text{otherwise} \end{cases}$$

for some non-decreasing correspondence $\Gamma_i(m_{-i}) \subseteq \mathcal{X}_i \setminus \{\emptyset\}$: $\Gamma_i(m'_{-i}) \subseteq \Gamma_i(m_{-i})$ with $m'_{-i} \subseteq m_{-i}$. Accompanied by the extension of the message space, I impose the following restriction to constraint correspondences. A profile $(\Gamma_{-i}(\cdot), \Gamma_i(\cdot))$ of constraint correspondences is said to *satisfy the independence of irrelevant contracts* if for each agent i and a pair of message profiles (m_{-i}, m_i) and (m'_{-i}, m'_i), if (1) $B_i \in m'_i \subseteq m_i$ and $B_i \in \Gamma_i(m_{-i})$, and (2) $\Gamma_i(m'_{-i}) \subseteq \Gamma_i(m_{-i})$, then $B_i \in \Gamma_i(m'_{-i})$. The outcome function is defined by

$$g(m_{-i}, m_i) = \bigcup_{g_i(m_{-i}, m_i) \neq \emptyset} g_i(m_{-i}, m_i).$$

The ordered pair (M, g) is called a *demand revelation mechanism with general offers* if the associated constraint correspondences are non-decreasing and independent of irrelevant contracts. The attainable set of each agent i is given by $\Gamma_i(m_{-i}) \cup \{\emptyset\}$ as in the previous subsection. The following theorem examines effects of the flexibility of announcements due to the extension of the message space.

Theorem 2. *Any subcorrespondence of the Pareto choice rule is not Nash-implementable by any demand revelation mechanism with general offers.*

Proof. See Appendix 2. □

Corollary 2. *The core correspondence φ_C is not Nash-implementable by any demand revelation mechanism with general offers.*

Appendix 1: Proof of Theorem 1

Proof. I shall show that a necessary condition for Nash-implementation of any subcorrespondence of the Pareto choice rule by a demand revelation mechanism will not be satisfied over the weakly substitutable preference domain. A stronger notion of Maskin monotonicity appeared in Sjöström (1996) plays a crucial role.[6] I introduce some notions. For each allocation $X' \in A$ and each preference relation $R_i \in \mathcal{R}_i$, define agent i's lower contour set at X' under R_i by $L_i(X', R_i) = \{B_i \in \mathcal{X}_i \mid C_{R_i}(X') R_i B_i\}$. Define $\Lambda_i^{\varphi}(X') = \bigcap_{R \in \varphi^{-1}(X')} L_i(X', R_i)$, where $\varphi^{-1}(X') = \{R \in \mathcal{R} \mid X' \in \varphi(R)\}$ is the contour set of φ at X'. Then, a choice rule φ is *strongly monotonic* if for every allocation $X' \in A$ and every $R' \in \mathcal{R}$, if $X' \in \varphi(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} \varphi(R)$ and $\Lambda_i^{\varphi}(X') \subseteq L_i(X', R_i')$ for every agent i, then $X' \in \varphi(R')$.[7]

[6] Sjöström (1996) analyzes exchange economies with *perfectly divisible goods*. The model of this note can be considered as an exchange economy with *indivisible goods*.

[7] Strong monotonicity implies Maskin monotonicity. A choice rule φ is *Maskin monotonic* if for every allocation $X' \in A$ and every $(R, R') \in \mathcal{R} \times \mathcal{R}$, if $X' \in \varphi(R)$ and $L_i(X', R_i) \subseteq L_i(X', R_i')$ for every agent i, then $X' \in \varphi(R')$. See Maskin and Sjöström (2002) for further arguments about Maskin monotonicity.

The proof consists of two steps.

Step 1. *If a choice rule φ is Nash-implementable by a demand revelation mechanism with final offers, then φ is strongly monotonic.*

Proof of Step 1. Consider any $R' \in \mathcal{R}$. Suppose that $X' \in \varphi(\mathcal{R})$ and $\Lambda_i^{\varphi}(X') \subseteq L_i(X', R_i')$ for every agent i. I shall show that $X' \in \varphi(R')$. By the hypothesis, there exists a demand revelation mechanism with final offers (M, g) that implements φ in Nash equilibrium. Since $\varphi(R') = g(\mathcal{N}_{(M,g)}(R'))$, it suffices to find some equilibrium message profile $m^* \in \mathcal{N}_{(M,g)}(R')$ such that $g(m^*) = X'$. Set $m_i^* = C_{R_i}(X')$ for every agent i. I need some preparation before proving the requirements of such message profile $m^* = (m_{-i}^*, m_i^*)$. Consider any $R \in \varphi^{-1}(X')$. Since $X' \in \varphi(R) = g(\mathcal{N}_{(M,g)}(R))$, it follows that $X' = g(m_{-i}^R, m_i^R)$ for some $(m_{-i}^R, m_i^R) \in \mathcal{N}_{(M,g)}(R)$. Then, $\{g_i(m_{-i}^R, \hat{m}_i) \mid \hat{m}_i \in M_i\} \subseteq L_i(X', R_i)$ holds for every agent i.

Claim 1. *$m_i^R \, I_i \, m_i^*$ for every agent i. Moreover, $m_i^R = m_i^*$ for every agent i involved in X'.*

Proof of Claim 1. Recollect that either $g_i(m_{-i}^R, m_i^R) = m_i^R \neq \emptyset$ or $g_i(m_{-i}^R, m_i^R) = \emptyset$ by definition of the mechanism with final offers. Consider first any agent i involved in X'. It is always possible to be unmatched by submitting $m_i = \emptyset$ if he prefers the null contract to $g_i(m_{-i}^R, m_i^R) = m_i^R$. Therefore, it must be the case that $C_{R_i}(X') = g_i(m_{-i}^R, m_i^R) = m_i^R$, and hence $m_i^R = m_i^*$. Of course, this implies that $m_i^R \, I_i \, m_i^*$.

Each agent i outside X' is assigned to the null contract, that is, $g_i(m_{-i}^R, m_i^R) = \emptyset$, and so $m_i^* = C_{R_i}(X') = \emptyset$. It remains to show that $m_i^R \, I_i \, \emptyset$ for any agent i outside X'. By submitting the null contract $\emptyset$, the feasibility requirement such as $\emptyset \notin \Gamma_i(m_{-i}^R)$ yields that he must be unmatched. Therefore, $m_i^R \, I_i \, \emptyset$, and hence $m_i^R \, I_i \, m_i^*$.[8] This establishes the claim.

In what follows, without loss of generality, I may assume that $m_i^R = m_i'$ for every agent i.

Claim 2. $g(m_{-i}^*, m_i^*) = X'$.

Proof of Claim 2. The assertion is immediate from the fact that $g(m_{-i}^R, m_i^R) = X'$, together with $m_i^R = m_i^*$ for every agent i. This establishes the claim.

Claim 3. $(m_{-i}^*, m_i^*) \in \mathcal{N}_{(M,g)}(R')$.

Proof of Claim 3. Notice that $\{g_i(m_{-i}^*, \hat{m}_i) \mid \hat{m}_i \in M_i\} = \{g_i(m_{-i}^R, \hat{m}_i) \mid \hat{m}_i \in M_i\} \subseteq L_i(X', R_i)$, where the equality follows from Claim 1. Since $R \in \varphi^{-1}(X')$ was arbitrary, it follows that $\{g_i(m_{-i}^*, \hat{m}_i) \mid \hat{m}_i \in M_i\} \subseteq \bigcap_{R \in \varphi^{-1}(X')} L_i(X', R_i) = \Lambda_i^{\varphi}(X') \subseteq L_i(X', R_i')$, where the last inclusion follows from the hypothesis. Hence, $\{g_i(m_{-i}^*, \hat{m}_i) \mid \hat{m}_i \in M_i\} \subseteq L_i(g(m_{-i}^*, m_i^*), R_i')$ for every agent i. This establishes the claim.

Since the mechanism (M, g) implements φ in Nash equilibrium, it follows from Claims 2 and 3 that $X' = g(m_{-i}^*, m_i^*) \in \varphi(R')$. This establishes the step.

Step 2. *Any subcorrespondence of the Pareto choice rule is not strongly monotonic.*

[8] His best response m_i^R to m_{-i}^R is not necessarily the case that $m_i^R = \emptyset$ because $g_i(m_{-i}^R, m_i^R) = \emptyset$ is possible whenever $m_i^R \notin \Gamma_i(m_{-i}^R)$. This is the reason for which I do not claim the equivalence between m_i^R and m_i' for every agent i outside X'.

Proof of Step 2. Consider any $\varphi \subseteq \varphi_{PE}$. Consider the economy with $D = \{d_1, d_2\}$ and $H = \{h\}$. Let $X = \{x, y, z\}$ with $x_D = d_1 = y_D$ and $z_D = d_2$. Then,

$$\mathcal{X}_{d_1} = \{\emptyset, \{x\}, \{y\}\},\ \mathcal{X}_{d_2} = \{\emptyset, \{z\}\}, \text{and}$$
$$\mathcal{X}_h = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, z\}, \{y, z\}\}.$$

Notice that $\{x, y\} \notin \mathcal{X}_h$. Let $X' = \{x, z\}$. Define the following preference profiles $R = (R_{d_1}, R_{d_2}, R_h)$ and $R' = (R_{d_1}', R_{d_2}, R_h)$, where:

R_{d_1}	R_{d_2}	R_h	R_{d_1}'
$\{x\}$	$\{z\}$	$\{y, z\}$	$\{y\}$
$\{y\}$	$\emptyset$	$\{x, z\}$	$\{x\}$
$\emptyset$		$\{y\}$	$\emptyset$
		$\{x\}$	
		$\{z\}$	
		$\emptyset$	

These preference profiles R and R' are weakly substitutable indeed.

Claim 4. $X' \in \varphi(R)$ *but* $X' \notin \varphi(R')$.

Proof of Claim 4. Firstly, the set of balanced allocations is given by $\{\{\emptyset\}, \{x\}, \{y\}, \{z\}, \{x, z\}, \{y, z\}\}$. With respect to the preference profile R, it can be shown that the set of Pareto efficient allocations coincides with the core. More precisely, $\varphi_{PE}(R) = \{\{x, z\}, \{y, z\}\} = \varphi_C(R)$. Since any choice rule that is Nash-implementable by a demand revelation mechanism with final offers is strongly monotonic, it is Maskin monotonic. By Theorem 1 in Kara and Sönmez (1996), any Maskin monotonic subcorrespondence φ of the Pareto choice rules must be a supersolution of φ_C, that is, $\varphi_C \subseteq \varphi$.[9] Therefore, $\varphi_C(R) \subseteq \varphi(R) \subseteq \varphi_{PE}(R)$ holds. Since $\varphi_{PE}(R) = \varphi_C(R)$, it follows that $\varphi(R) = \{\{x,z\},\{y,z\}\}$, and hence $X' = \{x,z\} \in \varphi(R)$. Moreover, $X' \notin \varphi(R')$ because the allocation $\{y,z\}$ Pareto dominates $X' = \{x,z\}$ under R'. This establishes the claim.

It remains to show that $\Lambda_i^{\varphi}(X') \subseteq L_i(X', R')$ for every agent i. Regarding agent d_1, the assertion is trivial because $\Lambda_{d_1}^{\varphi}(X') \subseteq \mathcal{X}_{d_1}$ by definition, and $L_{d_1}(X', R'_{d_1}) = \{\emptyset, \{x\}, \{y\}\} = \mathcal{X}_{d_1}$. Similar to agent d_2. For agent h, notice that $L_h(X', R'_h) = \mathcal{X}_h \setminus \{\{y,z\}\}$, where $R'_h = R_h$ in the preference profile R'. The fact that $\{y,z\} \notin L_h(X', R_h)$, together with $R \in \varphi^{-1}(X')$, implies that $\Lambda_h^{\varphi}(X') \subseteq \mathcal{X}_h \setminus \{\{y,z\}\}$. Therefore, $\Lambda_h^{\varphi}(X') \subseteq L_h(X', R'_h)$. Hence, I conclude that the subcorrespondence φ is not strongly monotonic. This establishes the step.

Steps 1 and 2 establish the theorem. □

Appendix 2: Proof of Theorem 2

Proof. Since strong monotonicity of any subcorrespondence of the Pareto choice rule is not satisfied over the weakly substitutable preferences by Step 2 in the proof of Theorem 1, it suffices to show that if a subcorrespondence $\varphi \subseteq \varphi_{PE}$ is Nash-implementable by a demand revelation mechanism with general offers, then φ is strongly monotonic. Consider any $R' \in \mathcal{R}$. Suppose that $X' \in \varphi(\mathcal{R})$ and $\Lambda_i^{\varphi}(X') \subseteq L_i(X', R'_i)$ for every agent i. I shall show that $X' \in \varphi(R')$. By the hypothesis, there exists a demand revelation mechanism with general offers (M, g) that implements φ in Nash equilibrium. Let $R \in \varphi^{-1}(X')$. Denote an equilibrium message profile by $(m_{-i}^R, m_i^R) \in \mathcal{N}_{(M,g)}(R)$ such that $X' = g(m_{-i}^R, m_i^R)$. Set $m_i^* = \{C_{R_i}(X')\}$ for each agent i.

Claim 1. *$m_i^* \subseteq m_i^R$ for every agent i involved in X'.*

Proof of Claim 1. Consider any agent i involved in X'. Since $g_i(m_{-i}^R, m_i^R) = C_{R_i}(X')$, it must be the case that $C_{R_i}(X') \in m_i^R$, and hence $m_i^* = \{C_{R_i}(X')\} \subseteq m_i^R$. This establishes the claim.

The following claim asserts that any agent who is unmatched in X' has no chance to obtain any contract but the null contract.

Claim 2. *For every agent i outside X', there is no $m_i \in M_i$ such that $(m_{-i}^R, m_i) \in \mathcal{N}_{(M,g)}(R)$ and $B_i \in \Gamma_i(m_{-i}^R)$ for some $B_i \in m_i$.*

Proof of Claim 2. Suppose, by way of contradiction, that there is a message $m_i \in M_i$ such that $(m_{-i}^R, m_i) \in \mathcal{N}_{(M,g)}(R)$ and $B_i \in \Gamma_i(m_{-i}^R)$ for some $B_i \in m_i$. Then, $g_i(m_{-i}^R, m_i) = B_i \neq \emptyset$ because $\emptyset \notin \Gamma_i(m_{-i}^R)$. On the other hand, since both messages m_i^R and m_i are best responses to m_{-i}^R by the hypothesis, it follows from antisymmetry of his preference R_i that $g_i(m_{-i}^R, m_i^R) = g_i(m_{-i}^R, m_i)$. This yields that $g_i(m_{-i}^R, m_i) = \emptyset$, a contradiction. This establishes the claim.

Regarding agent i outside X', without loss of generality, I may assume that $m_i^R = \{\emptyset\}$ by Claim 2. Since $m_i^* = \{C_{R_i}(X')\} = \{\emptyset\}$,

[9] Kara and Sönmez (1996) show that the core correspondence is the minimal Maskin monotonic subsolution of the Pareto and individually rational rule in one-to-one matching problems without contracts. Their result carries over to many-to-one matching problems with contracts. In addition, in this note, the individually rationality of allocations is incorporated in the balancedness of allocations.

it follows that $m_i^* \subseteq m_i^R$ for every agent i outside X'. Thus, I assume that $m_i^* \subseteq m_i^R$ for every agent i in what follows.

Claim 3. $g(m_{-i}^*, m_i^*) = X'$.

Proof of Claim 3. Notice that for every agent i involved in X', $C_{R_i}(X') \in m_i^* \subseteq m_i^R$ and $C_{R_i}(X') \in \Gamma_i(m_{-i}^R)$. In addition, since $\Gamma_i(\cdot)$ is non-decreasing, it follows that $\Gamma_i(m_{-i}^*) \subseteq \Gamma_i(m_{-i}^R)$. The independence of irrelevant contracts of the constraint correspondences yields that $C_{R_i}(X') \in \Gamma_i(m_{-i}^*)$, and so $g_i(m_{-i}^*, m_i^*) = C_{R_i}(X')$ for every agent i involved in X'. In contrast, for every agent i outside X', $g_i(m_{-i}^*, m_i^*) = \emptyset$ because of $m_i^* = \{\emptyset\}$. Thus, $g(m_i^*, m_i^*) = X'$. This establishes the claim.

Claim 4. $(m_{-i}^*, m_i^*) \in \mathcal{N}_{(M,g)}(R')$.

Proof of Claim 4. Similar to the proof of Claim 3 in the proof of Theorem 1.

Since the mechanism (M, g) implements φ in Nash equilibrium, it follows from Claims 3 and 4 that $X' = g(m_{-i}^*, m_i^*) \in \varphi(R')$. This establishes strong monotonicity of φ. By Step 2 in the proof of Theorem 1, the choice rule φ violates strong monotonicity over the weakly substitutable preference domain. This establishes the theorem. □

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