

Theoretical Study of Vibration of a Bowed String

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1. Introduction

Vibration of a bowed string, of musical instruments like a violin, has been considered to be a typical example of self-excited vibration of a continuous body under solid friction. However, until several years ago, we did not have any satisfactory theory that treats it as a problem of self-excited vibration as such. The celebrated formula of H. von Helmholtz¹⁾ indicates that the excited mode of vibration is identical with one of the various modes of natural vibration of a string with two fixed ends, and that one period of vibration consists of two intervals, i. e., a stick interval and a slip interval. During the former, the bowed point of the string follows the motion of the bow, while during the latter the bowed point recoils back being freed from the bow. The ratio of the stick interval to the whole period is equal to the ratio of the distance between the bowed point and the remoter end of the string to its whole length.

There had been until then comparatively small number of theoretical studies²⁾³⁾ on the steady vibration of a bowed string, in contrast to a great number of experimental studies⁴⁾⁵⁾ that had been made especially after the advent of electronic measurement devices in 1930's. Moreover, almost all of the then existing theories start with the assumption of no damping and regard the Helmholtz' theory as being firmly established. We note that these theories have several unrealistic aspects: strictly speaking, these models would not allow a violin to emit any sound, and once the steady state of vibration is established, no bowing force would be required to maintain the vibration except for the constant statically deflecting force in the stick interval and the constant dynamic frictional force corresponding to the constant relative velocity in the slip interval. These two forces are of equal magnitude and the bowing force is entirely constant during the whole interval.

However, in the real situation where the vibration of a bowed string is subject to various types of damping and stiffness, e. g., air resistance, internal friction, sound energy loss through the bridge, bending stiffness and torsional compliance, the bowing force would not be constant. In order to compensate for the various energy losses mentioned above, the bowing force during the stick interval should be greater in

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the mean than that during the slip interval. Therefore, we have to determine in advance these bowing forces, namely that during the stick interval and that during slip interval, to construct an adequate theory of the steady vibration of a bowed string subject to various effects of damping and stiffness.

In the year 1975, one of the authors found that a Fourier series method utilizing a series transformation is applicable to the exact analysis of vibration of a bowed string; this method had been advocated by him in 1957⁶⁾⁷⁾⁸⁾ in order to obtain an exact solution of nonlinear vibration in a system having piecewise-linear characteristics. By applying this method, we are able to take into consideration all the effects above mentioned of damping and stiffness and to treat the problem properly in the scope of self-excited vibration⁹⁾¹⁰⁾¹¹⁾¹²⁾¹³⁾.

Meanwhile, a new theory of vibration of a bowed string was constructed by an integral equation method, said to be powerful enough to deal with not only steady periodic vibrations but also transient vibrations with all of the above-mentioned effects of damping and stiffness took into account¹⁴⁾¹⁵⁾.

The philosophy underlying this method, as to the steady vibrations, is the same as that of the authors', namely, on the one hand, the velocity of the bowed point is considered to be a frequency response to the bowing force as an input to a linear system representing a violin as a whole, and, on the other hand the bowing force and the velocity of the bowed point must be related to each other according to the frictional force characteristic between the string and the bow. These two conditions together yield the key equations for the solution.

2. Equations of Motion

A string of a violin is stretched under a tension T between two fixed points A and D while going over the bridge at B and is bowed by a bow moving with a constant velocity v_0 perpendicularly to the string at the bowed point C (Fig. 1). The breadth of the bow is assumed to be nil, namely the point C is a mathematical point. The length of the vibrating half AB of the string is l and the distances BC and AC between the bowed point and the two ends are ξ and $\bar{\xi}$ respectively. We assume that $\bar{\xi} \geq \xi$.

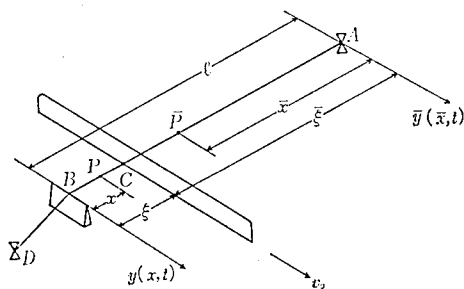


Fig. 1 Bowed String over Bridge and Bow

We suppose that the string vibrates in the plane determined by the straight line AB and the velocity vector $v_0^{*3)}$. We divide the whole vibrating half of the string into two parts, namely the left part BC and the right part AC of the bowed point C . An arbitrary point P on the former is specified by its distance x from B and an arbitrary point \bar{P} on the latter is specified by its distance \bar{x} from A . If we denote the displacement at time t of P on the left part and that of \bar{P} on the right part by $y(x, t)$ and $\bar{y}(\bar{x}, t)$ respectively, the equations of motion for these parts will be

$$\left. \begin{aligned} \partial^2 y / \partial t^2 + 2c \partial y / \partial t + (EI/\mu) \partial^4 y / \partial x^4 + (\bar{\tau} EI/\mu) \partial^5 y / \partial x^4 \partial t &= a^2 \partial^2 y / \partial x^2 \\ \partial^2 \bar{y} / \partial t^2 + 2c \partial \bar{y} / \partial t + (EI/\mu) \partial^4 \bar{y} / \partial \bar{x}^4 + (\bar{\tau} EI/\mu) \partial^5 \bar{y} / \partial \bar{x}^4 \partial t &= a^2 \partial^2 \bar{y} / \partial \bar{x}^2 \end{aligned} \right\} \quad (1)$$

where $a = \sqrt{T/\mu}$ is the propagation velocity of the lateral wave in case of no damping and no stiffness, and T being the tension, μ being the line density of the string, $2c$ is the damping coefficient of air resistance. The force of air resistance acting on a small element dx of the string with the velocity $\partial y / \partial t$ is given by $2c\mu dx \partial y / \partial t$. And EI is the bending stiffness, E being Young's modulus and I being the moment of inertia of cross sectional area of the string, and $\bar{\tau}$ is the coefficient of internal friction in the string.

After transients of short duration, the vibration becomes steady and periodic. Here, in this paper, we consider only such type of this steady vibration whose period is composed of one stick interval and one slip interval only and designate its circular frequency by ω , which is not necessarily equal to the lowest natural circular frequency $\nu = \pi a/l$ of the string with no damping and no stiffness. This circular frequency ω of the resulting vibration is not known at the outset of our analysis and will be determined afterwards. Utilizing this ω , we introduce a dimensionless time $\theta = \omega t$, whose origin coincides with the middle instant of the stick interval. Measuring by this time, lengths of the period, of the stick interval and of the slip interval become 2π , θ_0 and $\phi_0 = 2\pi - \theta_0$ respectively. For convenience of description, we introduce another dimensionless time $\psi = \theta - \pi$ whose origin coincides with the middle instant of the slip interval.

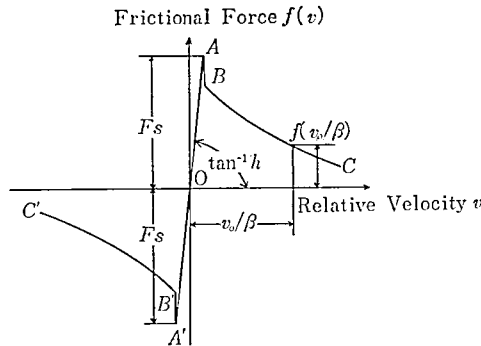


Fig. 2 Frictional Force Characteristic

*3) If the string is performing a whirling vibration, we consider the component of its motion in this plane.

The characteristic of the frictional force acting between the bow and the string is illustrated in Fig. 2. The straight line segment AOA' represents the static frictional force and the two curves BC and $B'C'$ concave upward represent the dynamic frictional forces. We may have some jumps AB and $A'B'$ between the static and the dynamic regions.

The bowing force $B(\theta)$, namely the frictional force exerted by the bow on the bowed point of the string, becomes also a periodic function of θ with the period 2π and can be considered to be composed of two components, namely the constant static deflecting force equal to the dynamic frictional force $f(v_0/\beta)$ corresponding to the mean relative velocity v_0/β during the slip interval and the additional bowing force $G(\theta)$, where β represents the slip interval ratio $\phi_0/(2\pi)$. Then we have

$$B(\theta) = f(v_0/\beta) + G(\theta) \quad (2)$$

Similarly, the displacements $y(x, t)$ and $\bar{y}(\bar{x}, t)$ of the string can also be divided into two parts as follows,

$$\left. \begin{aligned} y(x, t) &= y_s(x) + \eta(x, t) \\ \bar{y}(\bar{x}, t) &= \bar{y}_s(\bar{x}) + \bar{\eta}(\bar{x}, t) \end{aligned} \right\} \quad (3)$$

where $y_s(x)$ and $\bar{y}_s(\bar{x})$ are the static deflections due to the constant force $f(v_0/\beta)$ at the bowed point C and $\eta(x, t)$ and $\bar{\eta}(\bar{x}, t)$ are the vibrational displacements due to the additional bowing force $G(\theta)$ of the left and right halves of the string. The displacements of static deflection, $y_s(x)$ and $\bar{y}_s(\bar{x})$, are given by

$$\left. \begin{aligned} y_s(x) &= (\xi(x + T/k_0)/(l + T/k_0))f(v_0/\beta)/T \\ \bar{y}_s(\bar{x}) &= (\bar{x}(\xi + T/k_0)/(l + T/k_0))f(v_0/\beta)/T \end{aligned} \right\} \quad (4)$$

where k_0 is the spring constant of the body at the bridge with respect to the deflecting force acting parallel to the bow velocity v_0 . The differential equations governing $\eta(x, t)$ and $\bar{\eta}(\bar{x}, t)$ are the same as those of (1) governing $y(x, t)$ and $\bar{y}(\bar{x}, t)$.

3. Boundary Conditions

As seen in the foregoing paragraph, the differential equations of motion are linear, but the boundary conditions are nonlinear and become discontinuous at the transition points between the stick and slip intervals.

At the right end, we have

$$\eta(0, t) = 0 \quad (5)$$

and at the left end, a force equal to

$$T\partial\eta/\partial x|_{x=+0} \quad (6)$$

is transmitted to the body through the bridge. The generalized coordinates describing the vibrational motion of the body considered as a vibrating system of n degree-of-freedom are denoted by $\phi_1, \phi_2, \dots, \phi_n$. Then we have

$$\eta(0, t) = r_1\phi_1 + r_2\phi_2 + \dots + r_n\phi_n \quad (7)$$

and

$$\left. \begin{aligned}
M_n' e^{-j\phi_n'} &= \left(\frac{l^2}{\xi \bar{\xi}} \right) \frac{1}{\sqrt{Z_n}} \frac{\left(\sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} \xi + s_n \frac{\bar{\omega}_n - j\bar{c}_n}{a} l \cos \frac{\bar{\omega}_n - j\bar{c}_n}{a} \xi \right) \sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} \bar{\xi}}{\left(\sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} l + s_n \frac{\bar{\omega}_n - j\bar{c}_n}{a} l \cos \frac{\bar{\omega}_n - j\bar{c}_n}{a} l \right)} \\
M_n'' e^{-j\phi_n''} &= \left(\frac{l^2}{\xi \bar{\xi}} \right) \frac{1}{\sqrt{Z_n}} \frac{\sinh \frac{\bar{\alpha}_n - j\bar{\beta}_n}{a} \xi \sinh \frac{\bar{\alpha}_n - j\bar{\beta}_n}{a} \bar{\xi}}{\frac{\bar{\alpha}_n - j\bar{\beta}_n}{a} l \sinh \frac{\bar{\alpha}_n - j\bar{\beta}_n}{a} l}
\end{aligned} \right\} \quad (17)$$

and

$$s_n = \left(\frac{T}{l} \right) \sum_{i=1}^n \frac{r_i^2 / a_i}{\nu_i^2 - n^2 \omega^2 + 2j\zeta_i \nu_i n \omega} \quad (18)$$

and $N_n e^{-j\phi_n(x)}$ and $\bar{N}_n e^{-j\phi_n(\bar{x})}$ can be obtained from the expressions (16) and (17), in which ξ is replaced by x in the case of the former and $\bar{\xi}$ is replaced by \bar{x} in the case of the latter respectively, except for ξ and $\bar{\xi}$ in the expression $(l^2/\xi\bar{\xi})$ and the quantities $\bar{\omega}_n, \bar{c}_n, \bar{\alpha}_n$ and $\bar{\beta}_n$ are given by

$$\left. \begin{aligned}
\frac{\bar{\omega}_n - j\bar{c}_n}{a} l \\
\frac{\bar{\alpha}_n - j\bar{\beta}_n}{a} l
\end{aligned} \right\} = \frac{2\pi}{\varepsilon} \left(\frac{\sqrt{Z_n} \mp 1}{2(1 + jn\Omega\tau)} \right)^{1/2} \quad (19)$$

and

$$Z_n = 1 + \varepsilon^2 (n^2 \Omega^2 - 2jn\Omega\gamma) (1 + jn\Omega\tau) \quad (n=0, 1, 2, \dots) \quad (20)$$

$$\gamma = c/\nu, \quad \varepsilon = (2\pi/l) \sqrt{EI/T}, \quad \tau = \bar{\tau}\nu \quad \text{and} \quad \Omega = \omega/\nu \quad (21)$$

5. Determination of Fourier Coefficients by means of Series Transformation Method

We perform the following series transformation

$$G'(\theta) = K'(\theta) + H'(\phi), \quad \phi = \theta - \pi \quad (22)$$

where $K'(\theta)$ and $H'(\phi)$ are Fourier expansions of the additional bowing force $G'(\theta)$, which are valid only in the stick interval and only in the slip interval respectively. They are written as follows,

$$\left. \begin{aligned}
K'(\theta) &= \sum_{m=0}^{\infty} (\alpha_{2m}' \cos(\mu_{2m}\theta) + \beta_{2m+1}' \sin(\mu_{2m+1}\theta)), \quad \text{for } -\frac{1}{2}\theta_0 < \theta < \frac{1}{2}\theta_0 \\
K'(\theta) &= 0, \quad \text{for } \frac{1}{2}\theta_0 < \theta < 2\pi - \frac{1}{2}\theta_0 \\
H'(\phi) &= \sum_{m=0}^{\infty} (\bar{\alpha}_{2m}' \cos(\bar{\mu}_{2m}\phi) + \bar{\beta}_{2m+1}' \sin(\bar{\mu}_{2m+1}\phi)), \quad \text{for } -\frac{1}{2}\phi_0 < \phi < \frac{1}{2}\phi_0 \\
H'(\phi) &= 0, \quad \text{for } \frac{1}{2}\phi_0 < \phi < 2\pi - \frac{1}{2}\phi_0
\end{aligned} \right\} \quad (23)$$

$$\left. \begin{aligned}
&\dots \\
&\dots
\end{aligned} \right\} \quad (24)$$

where

$$\mu_{2m} = 2m\pi/\theta_0, \quad \mu_{2m+1} = (2m+1)\pi/\theta_0, \quad \bar{\mu}_{2m} = 2m\pi/\phi_0 \quad \text{and} \quad \bar{\mu}_{2m+1} = (2m+1)\pi/\phi_0 \quad (25)$$

Thus, we adopt Fourier cosine series for the stick interval and for the slip interval.

The old Fourier coefficients a_0', a_n' and b_n' ($n=1, 2, \dots$) valid for the whole period can be calculated from the new coefficients $\alpha_{2m}', \beta_{2m+1}'$ ($m=0, 1, 2, \dots$) for the stick interval and $\bar{\alpha}_{2m}', \bar{\beta}_{2m+1}'$ ($m=0, 1, 2, \dots$) for the slip interval by applying the following transformation formulas

$$\left. \begin{aligned} a_0' &= (\theta_0/2\pi)\alpha_0' + (\phi_0/2\pi)\bar{\alpha}_0' \\ a_n' &= (2n/\pi) \sin(n\theta_0/2) \left\{ \sum_{m=0}^{\infty} (-1)^{m+1} \alpha_{2m}' / (\mu_{2m}^2 - n^2) \right. \\ &\quad \left. - \sum_{m=0}^{\infty} (-1)^{m+1} \bar{\alpha}_{2m}' / (\bar{\mu}_{2m}^2 - n^2) \right\} \\ b_n' &= (2n/\pi) \cos(n\theta_0/2) \left\{ \sum_{m=0}^{\infty} (-1)^m \beta_{2m+1}' / (\mu_{2m+1}^2 - n^2) \right. \\ &\quad \left. + \sum_{m=0}^{\infty} (-1)^m \bar{\beta}_{2m+1}' / (\bar{\mu}_{2m+1}^2 - n^2) \right\} \end{aligned} \right\} \quad (26)$$

Furthermore, we have Fourier expansions for $\cos(n\theta)$ and $\sin(n\theta)$ which are valid only in the stick interval as

$$\left. \begin{aligned} \cos(n\theta) &= \sin(n\theta_0/2) / (n\theta_0/2) + (4n/\theta_0) \sin(n\theta_0/2) \\ &\quad \times \sum_{k=1}^{\infty} (-1)^{k-1} \cos(\mu_{2k}\theta) / (\mu_{2k}^2 - n^2) \\ \sin(n\theta) &= (4n/\theta_0) \cos(n\theta_0/2) \sum_{k=0}^{\infty} (-1)^k \sin(\mu_{2k+1}\theta) / (\mu_{2k+1}^2 - n^2) \end{aligned} \right\} \quad (27)$$

and quite similar ones valid only in the slip interval.

The right half of the boundary condition (9) for the stick interval can be thus transformed into

$$\dot{\eta}_\xi' + (l^2/\xi\bar{\xi})\sigma_1 K'(\theta) = 1 - f(v_0/\beta) / (h v_0) \quad (28)$$

where σ_1 is given by

$$\sigma_1 = T/(\pi h a) \quad (29)$$

and $\dot{\eta}_\xi'$ is the velocity of the bowed point and is given by differentiating the expression (13) for η_ξ' with respect to time t , namely

$$\dot{\eta}_\xi' = \sum_{n=1}^{\infty} n \Omega M_n \{ (a_n' \sin \phi_n + b_n' \cos \phi_n) \cos(n\theta) - (a_n' \cos \phi_n - b_n' \sin \phi_n) \sin(n\theta) \} \quad (30)$$

Substituting the expression (30) for $\dot{\eta}_\xi'$ in the equation (28) and then introducing the expressions (26) for a_0', a_n' and b_n' ($n=1, 2, \dots$) and those (27) for $\cos(n\theta)$ and $\sin(n\theta)$ into the resulting equation, the left side of the equation (28) becomes a Fourier series having $\cos(\mu_{2k}\theta)$ and $\sin(\mu_{2k+1}\theta)$ ($k=0, 1, 2, \dots$) as its harmonic components. Taking harmonic balance between the left and the right sides of this equation, we obtain an infinite set of simultaneous equations for Fourier coefficients $\alpha_{2m}', \beta_{2m+1}', \bar{\alpha}_{2m}'$ and $\bar{\beta}_{2m+1}'$ ($m=0, 1, 2, \dots$) as an infinite number of unknowns, namely

$$\left. \begin{aligned}
& \sum_{m=0}^{\infty} \{ (C_{km} + (l^2/\xi\bar{\xi})\sigma_1(1+\delta_{k0})\delta_{km})\alpha_{2m}' + D_{km}\beta_{2m+1}' + \bar{H}_{km}\bar{\alpha}_{2m}' + \bar{I}_{km}\bar{\beta}_{2m+1}' \} \\
& \quad = 2(1-f(v_0/\bar{\beta})/(hv_0))\delta_{k0} \\
& \sum_{m=0}^{\infty} \{ E_{km}\alpha_{2m}' + (F_{km} + (l^2/\xi\bar{\xi})\sigma_1\delta_{km})\beta_{2m+1}' + \bar{J}_{km}\bar{\alpha}_{2m}' + \bar{K}_{km}\bar{\beta}_{2m+1}' \} = 0 \\
& \quad (k=0, 1, 2, \dots)
\end{aligned} \right\} \quad (31)$$

where δ_{km} is the Kronecker's delta notation, namely we have

$$\delta_{km}=1 \text{ for } k=m \text{ and } \delta_{km}=0 \text{ for } k \neq m \quad (32)$$

and

$$\left. \begin{aligned}
C_{km} &= \frac{4}{\pi\theta_0} \Omega(-1)^{k+m} \sum_{n=1}^{\infty} \frac{n^3(1-\cos n\theta_0)}{(\mu_{2k}^2-n^2)(\mu_{2m}^2-n^2)} M_n \sin \phi_n \\
D_{km} &= -E_{mk} = \frac{4}{\pi\theta_0} \Omega(-1)^{k+m+1} \sum_{n=1}^{\infty} \frac{n^3 \sin n\theta_0}{(\mu_{2k}^2-n^2)(\mu_{2m+1}^2-n^2)} M_n \cos \phi_n \\
F_{km} &= \frac{4}{\pi\theta_0} \Omega(-1)^{k+m} \sum_{n=1}^{\infty} \frac{n^3(1+\cos n\theta_0)}{(\mu_{2k+1}^2-n^2)(\mu_{2m+1}^2-n^2)} M_n \sin \phi_n \\
& \quad (k, m=0, 1, 2, \dots)
\end{aligned} \right\} \quad (33)$$

$$\left. \begin{aligned}
\bar{H}_{km} &= -\frac{4}{\pi\theta_0} \Omega(-1)^{k+m} \sum_{n=1}^{\infty} \frac{n^3(1-\cos n\theta_0)}{(\mu_{2k}^2-n^2)(\bar{\mu}_{2m}^2-n^2)} M_n \sin \phi_n \\
\bar{I}_{km} &= \frac{4}{\pi\theta_0} \Omega(-1)^{k+m+1} \sum_{n=1}^{\infty} \frac{n^3 \sin n\theta_0}{(\mu_{2k}^2-n^2)(\bar{\mu}_{2m+1}^2-n^2)} M_n \cos \phi_n \\
\bar{J}_{km} &= \frac{4}{\pi\theta_0} \Omega(-1)^{k+m+1} \sum_{n=1}^{\infty} \frac{n^3 \sin n\theta_0}{(\mu_{2k+1}^2-n^2)(\bar{\mu}_{2m}^2-n^2)} M_n \cos \phi_n \\
\bar{K}_{km} &= \frac{4}{\pi\theta_0} \Omega(-1)^{k+m} \sum_{n=1}^{\infty} \frac{n^3(1+\cos n\theta_0)}{(\mu_{2k+1}^2-n^2)(\bar{\mu}_{2m+1}^2-n^2)} M_n \sin \phi_n \\
& \quad (k, m=0, 1, 2, \dots)
\end{aligned} \right\} \quad (34)$$

and

$$\left. \begin{aligned}
\mu_{2k} &= \frac{2k\pi}{\theta_0}, \quad \mu_{2k+1} = \frac{(2k+1)\pi}{\theta_0} \\
\bar{\mu}_{2k} &= \frac{2k\pi}{\phi_0}, \quad \bar{\mu}_{2k+1} = \frac{(2k+1)\pi}{\phi_0} \\
& \quad (k=0, 1, 2, \dots)
\end{aligned} \right\} \quad (35)$$

Similarly, the right half of the boundary condition (10) for the slip interval can be transformed into

$$H'(\phi) = A'((1-\beta)/\beta + \dot{\eta}_\xi') + A''((1-\beta)/\beta + \dot{\eta}_\xi')^2 + \dots \quad (36)$$

where

$$A' = -(df/dv)_{v=v_0/\beta}(\xi\bar{\xi}/l^2)\pi a/T, \quad A'' = (d^2f/dv^2)_{v=v_0/\beta}(\xi\bar{\xi}/l^2)(\pi a/T)v_0, \dots \quad (37)$$

By the similar procedure to that for the stick interval, we obtain another infinite set of simultaneous, in general, nonlinear equations for $\alpha_{2m}', \beta_{2m+1}', \bar{\alpha}_{2m}'$ and $\bar{\beta}_{2m+1}'$ ($m=0, 1, 2, \dots$) as an infinite number of unknowns.

In the case of $A''=A'''=\dots=0$ namely in the case of piecewise-linear characteristic of frictional force, this set of equations becomes linear and can be written as

$$\left. \begin{aligned}
& \sum_{n=0}^{\infty} \{ -A' H_{km} \alpha_{2n}' - A' I_{km} \beta_{2n+1}' + \{ (1 + \delta_{k0}) \bar{C}_{km} - A' \bar{C}_{km} \} \bar{\alpha}_{2n}' \\
& \quad - A' \bar{D}_{km} \bar{\beta}_{2n+1}' \} = 2A' \delta_{k0} (1 - \beta) / \beta \\
& \sum_{n=0}^{\infty} \{ -A' J_{km} \alpha_{2n}' - A' K_{km} \beta_{2n+1}' - A' \bar{E}_{km} \bar{\alpha}_{2n}' + (\delta_{km} - A' \bar{F}_{km}) \bar{\beta}_{2n+1}' \} = 0 \\
& \quad (k=0, 1, 2, \dots)
\end{aligned} \right\} \quad (38)$$

where \bar{C}_{km} , \bar{D}_{km} , \bar{E}_{km} , \bar{F}_{km} and H_{km} , I_{km} , J_{km} , K_{km} can be obtained by performing the substitution $\theta_0 \rightleftharpoons \phi_0$ in the expressions (33) and (34) for C_{km} , D_{km} , E_{km} , F_{km} and \bar{H}_{km} , \bar{I}_{km} , \bar{J}_{km} , \bar{K}_{km} .

6. Effect of Torsional Compliance of String

The string is susceptible to torsional vibration under the variable bowing force $B(\theta)$. The equations of motion for the vibrational angle of torsion $\phi(x, t)$ and $\bar{\phi}(\bar{x}, t)$ of the cross sections on the left and on the right sides of the string become

$$\partial^2 \phi / \partial t^2 + 2f \partial \phi / \partial t = b^2 \partial^2 \phi / \partial x^2, \quad \partial^2 \bar{\phi} / \partial t^2 + 2f \partial \bar{\phi} / \partial t = b^2 \partial^2 \bar{\phi} / \partial \bar{x}^2 \quad (39)$$

where $2f$ is the damping coefficient, $b = \sqrt{G/\rho}$ the propagation velocity of the torsional wave, G the modulus of rigidity, ρ the density of the string material.

As there is no slipping at the contact point, the string lags behind the bow by the distance of its rolling on the bow, namely by $r\phi_\xi$, where ϕ_ξ is the angle of torsion of the cross section at the bowed point. This lag amounts to

$$(r\phi_\xi)' = (2T/(G\pi r^2)) \mathcal{R} \sum_{n=0}^{\infty} \bar{M}_n' (a_n' - j b_n') e^{j(n\omega t - \bar{\phi}_n')} \quad (40)$$

in dimensionless expression, where r is the radius of the string, and

$$\bar{M}_n' e^{-j\bar{\phi}_n'} = \frac{l^2}{\xi \bar{\xi}} \frac{\sin \frac{\bar{\omega}_n' - j\bar{c}_n'}{b} \xi \sin \frac{\bar{\omega}_n' - j\bar{c}_n'}{b} \bar{\xi}}{\frac{\bar{\omega}_n' - j\bar{c}_n'}{b} l \sin \frac{\bar{\omega}_n' - j\bar{c}_n'}{b} l} \quad (41)$$

$$\left. \begin{aligned}
& \frac{\bar{\omega}_n' l}{b} \\
& \frac{\bar{c}_n' l}{b}
\end{aligned} \right\} = \frac{a}{b} \frac{n\pi\Omega}{\sqrt{2}} \sqrt{\sqrt{1 + \frac{4\gamma'^2}{n^2\Omega^2}} \pm 1}, \quad \gamma' = \frac{f}{\nu}, \quad \nu = \frac{\pi a}{l} \quad (42)$$

In consequence, we need only to add to various coefficients C_{km} , D_{km} , E_{km} , F_{km} and \bar{H}_{km} , \bar{I}_{km} , \bar{J}_{km} , \bar{K}_{km} in the equation (31) the correction terms $\sigma_2 C_{km}'$, $\sigma_2 D_{km}'$, $\sigma_2 E_{km}'$, $\sigma_2 F_{km}'$ and $\sigma_2 \bar{H}_{km}'$, $\sigma_2 \bar{I}_{km}'$, $\sigma_2 \bar{J}_{km}'$, $\sigma_2 \bar{K}_{km}'$ and to various coefficients in the equation (38) similar corrections, where

$$\sigma_2 = 2T/(G\pi r^2) = 2a^2/b^2$$

and C_{km}' , D_{km}' , E_{km}' , F_{km}' and \bar{H}_{km}' , \bar{I}_{km}' , \bar{J}_{km}' , \bar{K}_{km}' can be obtained by substituting \bar{M}_n' and $\bar{\phi}_n'$ for M_n and ϕ_n in the expressions (33) and (34). The similar coefficients in the equation (38) need to be similarly amended.

7. Effect of Compliance and Damping of Bowing Action

The action of bowing accompanies some spring action and some damping. The finiteness of the breadth of the bow in contrast to our assumption may also be con-

sidered as a source of some damping. We simply denote by K and C the spring constant and the damping coefficient of these effects of bowing action.

If we designate by δ the vibrational component of the lag of the bowed point behind the uniform forward motion of the bowing action, δ can be determined from the differential equation

$$C\dot{\delta} + K\delta = G(\theta) \quad (44)$$

Then, similarly to the foregoing case of the torsional compliance of the string, we only need to add to various coefficients C_{km}, D_{km}, \dots and $\bar{H}_{km}, \bar{I}_{km}, \dots$ etc., corrections $\sigma_3 C_{km}'', \sigma_3 D_{km}'', \dots$ and $\sigma_3 \bar{H}_{km}'', \sigma_3 \bar{I}_{km}'', \dots$ etc., where σ_3 is given as

$$\sigma_3 = Tl / (K\xi\bar{\xi}) \quad (45)$$

and $C_{km}'', D_{km}'', \dots$ and $\bar{H}_{km}'', \bar{I}_{km}'', \dots$ etc. can be obtained by substituting \bar{M}_n'' and $\bar{\phi}_n''$ given by

$$\bar{M}_n'' \cos \bar{\phi}_n'' = 1 / (1 + (2\xi n \Omega)^2), \quad \bar{M}_n'' \sin \bar{\phi}_n'' = 2\xi n \Omega / (1 + (2\xi n \Omega)^2) \quad (46)$$

$$\zeta = C\nu / (2K) \quad (47)$$

for M_n and ϕ_n in C_{km}, D_{km}, \dots and $\bar{H}_{km}, \bar{I}_{km}, \dots$ etc. of (33) and (34). Fortunately, this time we can obtain the infinite sums in (33) and (34) in closed forms.

8. Conditions of Capture and Release

When a violin is played by a player, we are given T, μ, γ, ϵ and τ as the parameters representing the tension, the line density, the air resistance, the stiffness and the internal friction of the string and ξ/l and v_0 as those representing the position and the velocity of the bow. We are also given $a_1, a_2, \dots, c_1, c_2, \dots, \nu_1, \nu_2, \dots$ and r_1, r_2, \dots as the parameters representing the sounding characteristics of the body responding to an excitation through the bridge. Also we are given b and γ' as the parameters characterizing the behavior for torsional vibration of the string and K and C as those for the bowing action. The frictional force characteristic between the string and the bow are also under our control.

Then, our indeterminate parameters remaining are only two in number, namely the frequency ratio Ω and the stick interval ratio $\theta_0/(2\pi)$.

Corresponding to these circumstances, we have further two conditions to be satisfied, namely the conditions of capture and release of the string by the bow. The instants of capture and release coincide with those of transition between the stick interval and the slip interval, where the frictional force attain the maximum static friction F_s . Accordingly, we have two conditions

$$G(\theta_0/2) = F_s - f(v_0/\beta) = G(-\theta_0/2) \quad (48)$$

Moreover, the law of static friction requires that the bowing force necessary to sustain the vibration must be always less than the maximum static friction F_s during the whole stick interval, namely we must have

$$|B(\theta)| < F_s, \quad -\theta_0/2 < \theta < \theta_0/2 \quad (49)$$

This condition is a condition of compatibility.

In the neighbourhood of the Helmholtz' mode $\Omega = 1.0$ and $\theta_0/(2\pi) = \bar{\xi}/l$, the

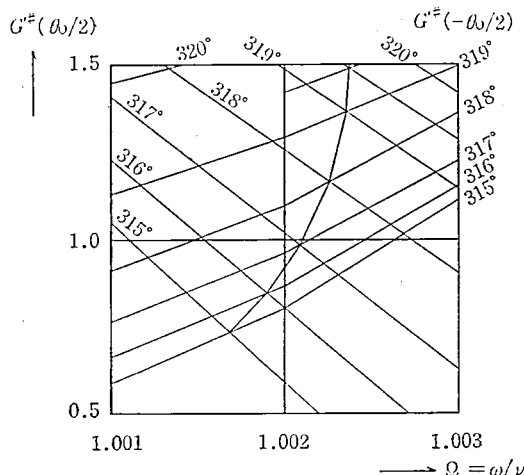


Fig. 3 Conditions of Release and Capture

vibrations of a stiff spring with $\varepsilon > 0$ have the tendency that with increasing Ω at the fixed θ_0 $G(\theta_0/2)$ increases and $G(-\theta_0/2)$ decreases, while with increasing θ_0 at the fixed Ω both $G(\theta_0/2)$ and $G(-\theta_0/2)$ increase, as shown in Fig. 3 for an example of a string with $\varepsilon=0.003$, $\tau=0.01$, $\gamma=0.001$, $b/a=\infty$ and $h=\infty$ and with the position of the bow $\xi/l=1/9$. We are able to determine on the locus curve $G(\theta_0/2)=G(-\theta_0/2)$ the desired combination of $\theta_0/(2\pi)$ and Ω for the given F_s and $f(v_0/\beta)$ by obtaining the point with the ordinate equal to $F_s - f(v_0/\beta)$.

9. Perfectly Flexible String Subject to Small Air Resistance

Coming back to the paragraph 5, if we have all $\bar{\alpha}_{2m}'$ and $\bar{\beta}_{2m+1}'$ ($m=0, 1, 2, \dots$) equal to zero and if α_{2m}' and β_{2m+1}' ($m=0, 1, 2, \dots$) obtained from solving the remaining equations (31) give a constant backward velocity $\dot{\eta}_s' = -(1-\beta)/\beta$ during the whole slip interval, this set of solutions also satisfies the boundary condition (36) of the slip interval, even when the frictional force characteristic is of essentially non-linear one as shown in Fig. 2, and gives an exact solution of self-excited vibration as such. The solutions obtained in the foregoing papers⁽⁹⁾⁽¹⁰⁾ starting with the assumption of piecewise-linear frictional force characteristic $A''=A'''=\dots=0$, with constant dynamic friction $A'=0$ belong to this class. Namely, if the stiffness of the string ε is zero and the damping ratio γ of air resistance is a small quantity of the first order and if we neglect small quantities of the order higher than the 2nd, our set of equations degenerates into one of

$$\left. \begin{aligned} \sum_{m=0}^{\infty} C_{km} \alpha_{2m}' &= 2(1-f(v_0/\beta)/(hv_0)) \delta_{k0} \\ (k=0, 1, 2, \dots) \end{aligned} \right\} \quad (50)$$

and especially for $\theta_0/(2\pi) \leq \xi/l$ and $h=\infty$, into one of

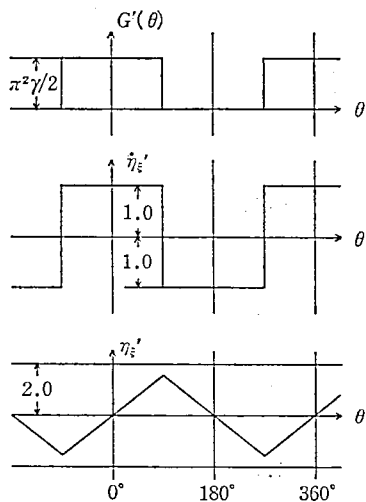


Fig. 4a Perfectly Flexible String
subject to Small Air
Resistance: $\varepsilon=\tau=\sigma_1=\sigma_2$
 $=\sigma_3=0$, $\gamma\rightarrow 0$, $\xi/l=1/2$,
 $\Omega=1.0$, $\theta_0=180^\circ$

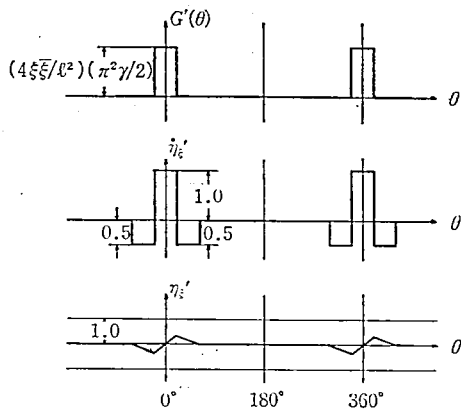


Fig. 4b Perfectly Flexible String
subject to Small Air
Resistance: $\varepsilon=\tau=\sigma_1=\sigma_2$
 $=\sigma_3=0$, $\gamma\rightarrow 0$, $\xi/l=1/9$,
 $\Omega=1.0$, $\theta_0=40^\circ$

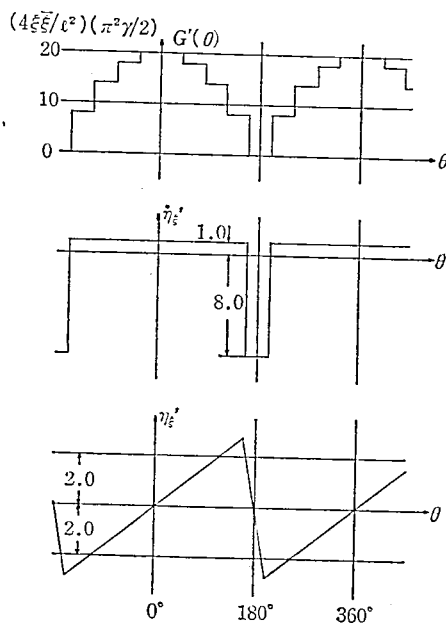


Fig. 4c Perfectly Flexible String subject to
Small Air Resistance: $\varepsilon=\tau=\sigma_1=\sigma_2$
 $=\sigma_3=0$, $\gamma\rightarrow 0$, $\xi/l=1/9$, $\Omega=1.0$,
 $\theta_0=320^\circ$

$$\left. \begin{aligned} C_{kk}\alpha_k' &= 2\delta_{k0} \quad \text{with} \quad C_{kk} = (2/(\pi\gamma^2)) (l^2/(4\xi\bar{\xi})) \\ &\quad (k=0, 1, 2, \dots) \end{aligned} \right\} \quad (51)$$

The solution in the latter case gives a constant additional bowing force during the stick interval equal to

$$G'(\theta) = (4\xi\bar{\xi}/l^2) (\pi\gamma^2/2) \quad (52)$$

Some examples of the solutions of these equations¹¹⁾ are shown in Fig. 4a for $\xi/l=1/2$, $\Omega=1.0$ and $\theta_0=180^\circ$, in Fig. 4b for $\xi/l=1/9$, $\Omega=1.0$ and $\theta_0=40^\circ$ and in Fig. 4c for $\xi/l=1/9$, $\Omega=1.0$ and $\theta_0=320^\circ$. The second example gives the backward velocity not uniform during the slip interval, so it is not a compatible solution except in the case of $A'=A''=\dots=0$. The third example can be obtained directly by solving (50), or in directly by superposing suitably chosen eight solutions of the second example. This superposition is effected by the following procedure. We divide the stick interval into eight equal parts and allocate k_1, k_2, k_3 and k_4 units of

$G^{\sharp'}(\theta)$

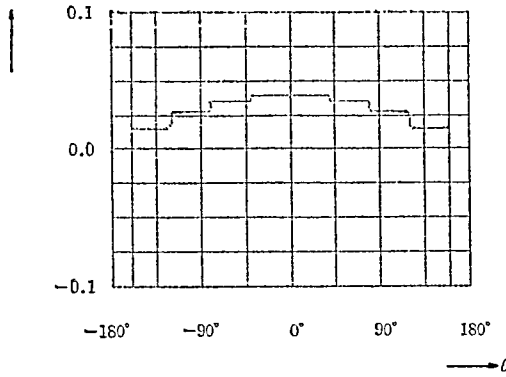


Fig. 5a Additional Bowing Force $G^{\sharp'}(\theta)$ for a String with $\gamma=0.001$, $\varepsilon=\tau=0$, $\sigma_1=0.05415$, $\sigma_2=0$, $\sigma_3=0.11536$, $\zeta=0.05$, $\xi/l=1/9$, $\Omega=1.0$, $\theta_0=320^\circ$

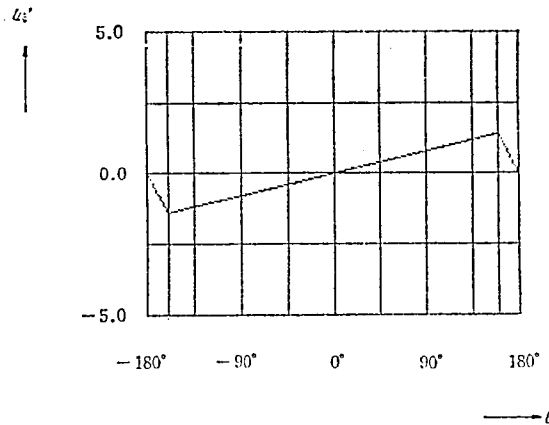


Fig. 5b Vibrational Displacement of Bowed Point of the same String as that in Fig. 5a

the solutions of the second example to each of these parts from the middle to both ends on both sides of the stick interval. The condition of the forward velocity during the stick interval being constant and equal to v_0 becomes

$$k_1 - (k_1 + k_2)/2 = k_2 - (k_1 + k_3)/2 = k_3 - (k_2 + k_4)/2 = k_4 - k_3/2 = 1 \quad (53)$$

yielding $k_1=20$, $k_2=18$, $k_3=14$ and $k_4=8$. This solution gives

$$\left. \begin{aligned} \alpha_{2m}' &= 0 \text{ for } m \text{ equal to a integral multiple of 4,} \\ \text{and for otherwise} \\ \alpha_0' &= 15L \text{ and } \alpha_{2m}' = (-1)^{m+1} L(8/m\pi) \cot(m\pi/8), \text{ where } L = 16\pi^2 \gamma / 81 \end{aligned} \right\} \quad (54)$$

and these α_{2m}' do actually satisfy the equations (50) for $h \rightarrow \infty$, $\xi/l = 1/9$, $\Omega = 1.0$ and $\theta_0 = 320^\circ$.

10. Numerical Examples of Typical Strings

Fig. 5a and Fig. 5b give the additional bowing force and the vibrational displacement of a perfectly flexible string namely with $\epsilon = \tau = 0$, subject to air resistance with $\gamma = 0.001$. The string has both ends fixed and is bowed at the bowed point $\xi = l/9$. The stick interval and the frequency are taken to be Helmholtz' values $\theta_0 = 320^\circ$ and $\Omega = 1.0$ and the compliance and the damping of bowing action are represented by $\sigma_3 = 0.11536$ and $\zeta = 0.05$, and the torsional compliance is neglected, namely $b/a = \infty$. The peculiar shape of the bowing force diagram coincides with the Raman's result²⁾ starting with a Helmholtz' mode of natural undamped vibration of a string.

The curve of vibration figure showing an instantaneous configuration of the vibrating string with damping must be composed of three segments, in contrast to two of that for Helmholtz' mode without damping, namely it must have a corner at the bowed point to give the additional bowing force of finite magnitude to overcome damping. These circumstances may be shown more clearly by adopting the unnaturally high value $\gamma = 0.05$ for air resistance. Fig. 6a and Fig. 6b show the additional

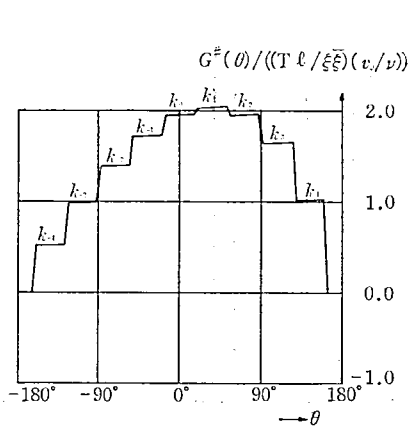


Fig. 6a Additional Bowing Force $G^{\#}(\theta)$ for a String with $\gamma=0.05$, $\epsilon=\tau=0$, $\sigma_1=\sigma_2=\sigma_3=0$, $\xi/l=1/10$, $\Omega=1.0$, $\theta_0=324^\circ$

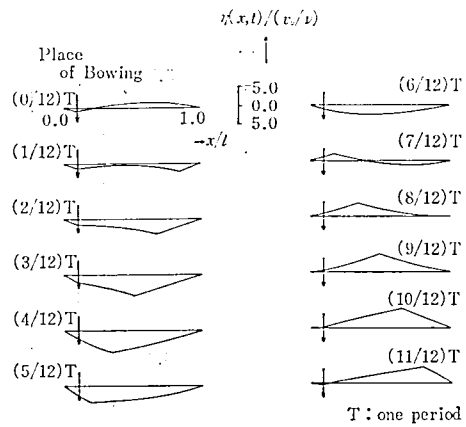


Fig. 6b Successive Configurations of the same Bowed String as that in Fig. 6a

bowing force and the successive configurations¹²⁾ at the intervals equal to one twelfth of one period of a string with $\xi/l=1/10$, $\theta_0=324^\circ$ and $\Omega=1.0$ and with $\varepsilon=\tau=\sigma_1=\sigma_2=\sigma_3=0$.

The computation for this example is performed by the approximate formulas of superposition mentioned in paragraph 9 in which the accuracy is by one more step improved. The additional bowing force diagrams in Fig. 5a and in Fig. 6a contradict with the compatibility condition (49) of release and capture. However, this point may not be made clear without taking into account the effect of bending stiffness, as illustrated by the following example.

Fig. 7a and Fig. 7b show the additional bowing force and the vibrational displacement of a stiff bowed string with $\varepsilon=0.003$ and $\tau=0.01$ subject to the air resistance of $\gamma=0.001$. The frequency ratio $\Omega=1.00075$ and the stick interval $\theta_0=320.72^\circ$ are so adjusted to satisfy the conditions of release and capture (48) with $G'(\theta_0/2)=G'(-\theta_0/2)=1.5^{13)}$. The remaining parameters are equal to those of the foregoing

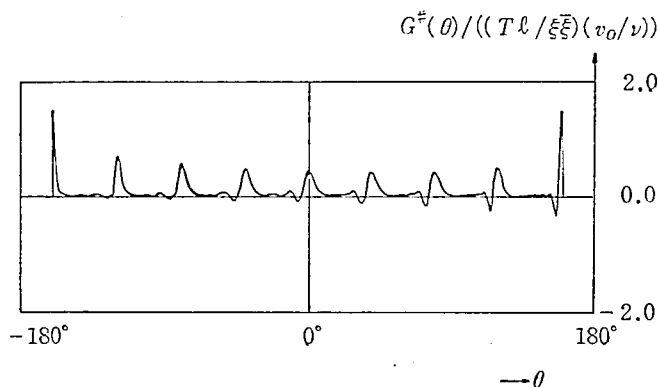


Fig. 7a Additional Bowing Force $G^{\sharp}(\theta)$ for a String with $\gamma=0.001$, $\varepsilon=0.003$, $\tau=0.01$, $\zeta=0.05$, $\sigma_2=0.11536$, $G^{\sharp}(-\theta_0/2)=G^{\sharp}(\theta_0/2)=1.5$, $\xi/l=1/9$, $\Omega=1.00075$, $\theta_0=320.72^\circ$, $h=\infty$

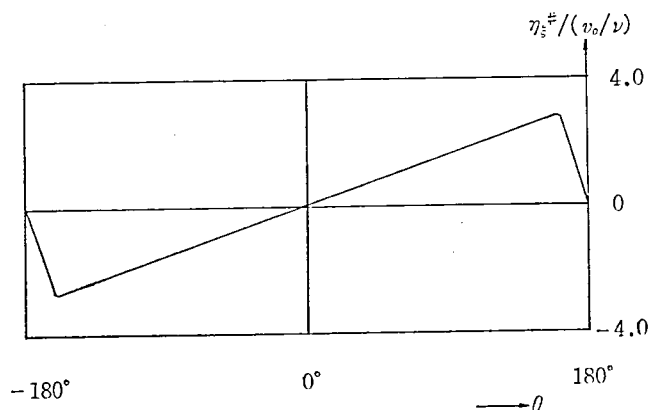


Fig. 7b Additional Vibrational Displacement η_{ξ}^{\sharp} for the same String as that in Fig. 7a

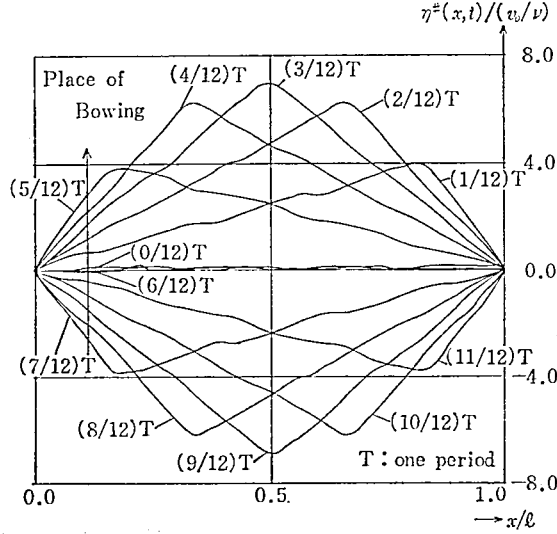


Fig. 7c Successive Configurations of a Bowed String with $\gamma=0.001$, $\epsilon=0.01$, $\tau=0.01$, $\sigma_1=\sigma_2=0$, $\sigma_3=0.11536$, $\zeta=0.05$, $\xi/l=1/9$, $\Omega=1.0$, $\theta_0=320^\circ$ (Statical Deflection due to Constant Pull $f(v_0/\beta)$ subtracted)

example, namely $A'=A''=\dots=0$, $b/a=\infty$, $\sigma_3=0.11536$ and $\zeta=0.05$. Here, the last crucial condition (49) of compatibility is also satisfied as seen in Fig. 7a.

The value $\epsilon=0.003$ is rather small one when considering that ϵ for an open E string of steel is equal to 0.0153. We have selected the value $\epsilon=0.003$ on account of the fact that various devices such as wound string are contrived to reduce the effect of stiffness so harmful to smooth playing and that there have been reported many experimental studies utilizing strings with small ϵ , for example, Krigar-Menzel's steel strings of diameter nearly equal to 0.1 mm and lengths $l=50$ cm and $l=80$ cm corresponding to $\epsilon=0.00366$ and $\epsilon=0.00143$ assuming the tuning to 440 Hz.

Fig. 7c shows successive configurations of a stiff bowed string with $\epsilon=0.01$, $\tau=0.01$, $\gamma=0.001$, $\sigma_1=\sigma_2=0$, $\sigma_3=0.11536$, $\zeta=0.05$, $\Omega=1.0$ and $\theta_0=320^\circ$.

Comparing Fig. 7a with Fig. 5a, we see that the presence of a small stiffness $\epsilon=0.003$ multiplies the maximum value of the additional bowing force almost 40 times. A bowed string is an essentially nonlinear vibrating system with an infinite number of natural frequencies. In the case of no stiffness and no damping, the ratio of the n -th harmonic natural frequency to the fundamental one is exactly equal to n , namely there is an infinite number of internal resonances. A bowed string with small damping vibrates with a frequency almost exactly equal to the fundamental natural frequency of free vibration. The required bowing force is not harmonic, namely has many higher harmonics owing to the strong nonlinearity of frictional force characteristic. These higher harmonic components of the bowing force are all in almost exact resonance to one of the natural frequencies of higher harmonic. This

situation of all modes being in resonance is the reason why there is required only a very small bowing force to maintain vibration of a perfectly flexible string subject to air resistance only.

The presence of even a small stiffness changes the situation completely, because the sequence of ratios of natural frequencies no longer forms that of positive integers and we have no internal resonance. Higher harmonic components of the bowing force have some detuning to any of the natural frequencies and must be much greater than those for the foregoing case of no stiffness.

11. Convergency and Lanczos' Smoothing Process

As seen in the foregoing paragraphs, we have to cope with the difficulty of solving an infinite set of simultaneous linear or nonlinear equations for Fourier expansion coefficients of the additional bowing force. The existence and uniqueness of this solution are not yet mathematically established. The only thing we can say now is that the solutions of the truncated set of equations seem to converge as number of equations increases and that these solutions satisfy the equations of motion exactly and the boundary conditions approximately.

Some of our functions to be expanded into Fourier series, for example, the bowing force and the velocity of the bowed point, are discontinuous. The Fourier series of a discontinuous function has a poor convergency and suffers from the Gibbs phenomena. Lanczos' process of smoothing lessens this phenomena and improves the convergency. We apply this process all the times when it is needed, and distinguish the resulting functions by adding the mark # to their functional names, as seen in the figures of foregoing paragraphs.

12. Conclusion

This paper is intended to be an interim report on our researches conducted by the authors and the co-worker K. Tenma of Kisarazu Technical College since several years ago, and is expected to continue for some future years, to obtain exact solutions for steady vibrations of a bowed string. The effects of air resistance, bending stiffness, internal friction, loss of sound energy through the bridge, torsional compliance of the string and damping and compliance of the bowing action are all taken into consideration, and the problem is treated properly in the scope of self-excited vibration as such.

The method of analysis is by means of a Fourier series solution utilizing series transformation for nonlinear vibrating system with piecewise-continuous characteristic. This method was originally introduced by one of the authors about twenty years ago to analyse steady forced vibration of a continuous body with piecewise-linear characteristic. It was only several years ago that the authors found this method being applicable to steady self-excited vibration in a continuous body with piecewise-continuous characteristic, for example, to vibration of a bowed string of violin.

The gist of analysis consists in solving an infinite set of simultaneous linear or nonlinear equations for Fourier expansion coefficients of the bowing force as an infinite number of unknowns. Nowadays, in contrast to the Raman's time when he rejected Fourier series method in favour of his velocity wave method²⁾, high speed digital computer with big memory has made Fourier series method more advantageous. However, solving one after another sets of, say, one hundred equations, whose coefficients are again obtained by a sum of, probably, three hundred terms, is not an easy task even today. Fortunately, the general environment is improving rapidly to encourage the authors very much.

Summarizing the main results obtained, the fundamental equations of motion have been derived with taking into consideration all the effects of damping and stiffness above mentioned. And we have established by theory following two facts. One is that the celebrated Helmholtz' mode is obtained as a limiting mode of vibration occurring in a bowed string with small stiffness subject to small air resistance as the stiffness and the resistance tend to zero. Next is that the presence of even a small stiffness increases the bowing force required to maintain vibration substantially. So the conditions of release and capture can not be discussed without consideration of stiffness. The Helmholtz' mode is not compatible with the law of static friction in relation to these conditions. Namely, the bowing force required to maintain this mode of vibration attains its maximum value at the middle of the stick interval. The pulse like peaks of the bowing force at the beginning and at the end of the stick interval, which are called by Schelleng⁴⁾ as "rabbit ears", are mainly due to the effect of stiffness.

The analysis of the effects of loss of sound energy through the bridge is perhaps one of the most fascinating problems in our research, but it is now barely started by the authors and is expected to be done as successfully as before.

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