On the Criterion of Fatigue Strength of a Round Bar Subjected to Combined Static and Repeated Bending and Torsion

*by Yuichi KAWADA**

Summary

A new criterion of fatigue strength under combined bending and torsion is proposed. The criterion is obtained, extending the internal friction criterion considered on the plane of maximum shearing stress, to the case of mean stress being applied. Derived design formula is simple, conforms well with experimental results by Gough, and includes several criteria already published as its special cases.

1. Introduction

In order to design machine parts subjected to combined stresses against fatigue failure, a criterion should be made clear. Several criteria are already published for fatigue strength of a round bar subjected to combined static and repeated bending and torsion, but the fatigue strength obtained by those criteria does not coincide strictly with the experimental results. In this paper, the author publishes a new criterion which is simple, includes many criteria already published as its special cases, and conforms well with experimental results.

2. Survey of criteria already published

Those criteria already published and included in the proposed criterion as its special cases are as follows. Denote principal stresses $\sigma_1, \sigma_2, \sigma_3$ and let the order of magnitude is $\sigma_1 > \sigma_2 > \sigma_3$. Let A, B and C are material constants.

2-1 Maximum principal stress criterion

$$\sigma_1 = A_1 \tag{1}$$

2-2 Maximum shearing stress criterion

$$\sigma_1 - \sigma_3 = A_1 \tag{2}$$

2-3 Modified maximum principal strain criterion

$$\sigma_1 - B_1\sigma_3 = A_1 \tag{3}$$

This criterion corresponds with Nishihara-Endo's criterion[1] for combined bending and torsion, and in this case, intermediate principal stress $\sigma_2 = 0$. B_1 is a material constant, not the Poisson's ratio and the criterion was named modified maximum principal strain criterion.

* 理工学部機械工学科教授　材料力学

2-4 Internal friction criterion (1)

$$\tau_\theta+B_2\sigma_{\theta M}=A_2 \tag{4}$$

This criterion corresponds with Nakanishi's criterion[2] for uniaxial stress states, and Findley's criterion[3] for plane-stress states. In the equation, τ_θ is the shearing stress amplitude on a plane inclined θ to the direction of the principal stress $\sigma_1, \sigma_{\theta M}$ is the maximum normal stress on the same plane inclined θ, and B_2 is a material constant which corresponds to the coefficient of friction. Assuming the plane of fracture is that, the left side of Eq. (4) becomes maximum, the direction of fracture plane θ is determined from the next equation.

$$d(\tau_\theta+B_2\sigma_{\theta M})/d\theta=0 \tag{5}$$

Considering the completely reversed stress state, τ_θ and $\sigma_{\theta M}$ are expressed by principal stresses as follows.

$$\tau_\theta=\frac{1}{2}(\sigma_1-\sigma_3)\ \sin 2\theta \tag{6}$$

$$\sigma_{\theta M}=\frac{1}{2}(\sigma_1+\sigma_3)+\frac{1}{2}(\sigma_1-\sigma_3)\ \cos 2\theta \tag{7}$$

Putting Eqs. (6) and (7) in Eq. (5), θ is determined, and putting this θ in Eqs. (4), (6) and (7), Eq. (3) is obtained. That is, internal friction criterion and modified maximum principal strain criterion are the same one.

2-5 Internal friction criterion (2)

$$\tau_{\pi/4}+B\sigma_{\pi/4}=A \tag{8}$$

This criterion corresponds with Matake's criterion[4] in the case of completely reversed bending and torsion. In the equation, $\tau_{\pi/4}$ is the shearing stress amplitude on the plane $\theta=\pi/4$, that is, on the maximum shearing stress plane, and $\sigma_{\pi/4}$ is the normal stress amplitude on the same plane. Then, Eq. (8) is expressed as Eq. (9).

$$\{(\sigma_1-\sigma_3)/2\}+\{B(\sigma_1+\sigma_3)/2\}=A \tag{9}$$

Transforming Eq. (9), Eq. (3) is obtained, and Matake's criterion is also the same with the modified maximum principal strain criterion.

2-6 Gough's empirical formula[5]

Gough published empirical formulas for completely reversed combined bending and torsion.

For cast iron and notched steel bars,

$$\left(\frac{\tau_a}{\tau_w}\right)^2+\left(\frac{\sigma_a}{\sigma_{wb}}\right)^2\left(\frac{\sigma_{wb}}{\tau_w}-1\right)+\frac{\sigma_a}{\sigma_{wb}}\left(2-\frac{\sigma_{wb}}{\tau_w}\right)=1 \tag{10}$$

For plain steel bars,

$$\left(\frac{\sigma_a}{\sigma_{wb}}\right)^2+\left(\frac{\tau_a}{\tau_w}\right)^2=1 \tag{11}$$

In these formulas, τ_a and σ_a are torsional and bending stress amplitude respectively, σ_{wb} the rotating bending fatigue limit, and τ_w the completely reversed torsional fatigue limit. Gough expressed in his paper that, Eq. (10) is derived from either Eq. (3) or Eq. (4). Matake expressed in the Appendix of his paper that, Eq. (10) is derived from Eq. (8). Putting $\sigma_{wb}/\tau_w=2$ in Eq. (10), Eq. (11) is derived.

Though 6 criteria are mentioned, sections 2-3 to 2-6 are cleared to be the same criteria for completely reversed combined bending and torsion. Putting $B_1=0$ in Eq. (3), Eq. (1) is obtained, and putting $B_1=1$ in Eq. (3), Eq. (2) is obtained, and sections 2-1 and 2-2 are also included in sections 2-3 to 2-6.

Material constants A_1 and B_1 are determined as follows. Applying Eq. (3) for completely reversed bending, and putting $\sigma_1=\sigma_{wb}$, $\sigma_3=0$, in Eq. (3), next relation is obtained.

$$A_1=\sigma_{wb} \tag{12}$$

Applying Eq. (3) for completely reversed torsion, and putting $\sigma_1=-\sigma_3=\tau_w$ in Eq. (3), we obtain

$$B_1=(\sigma_{wb}/\tau_w)-1 \tag{13}$$

According to Eq. (13), relation $\sigma_{wb}/\tau_w=1$ is obtained for $B_1=0$, and $\sigma_{wb}/\tau_w=2$ for $B_1=1$.

3. A new criterion of fatigue strength under combined static and repeated bending and torsion

Matake's internal friction criterion of section 2-5 was extended to the criterion of fatigue strength with mean stresses. Matake's criterion was that, fatigue failure occurs when maximum shearing stress amplitude attains a material constant A, but the material constant A is diminished if normal stress amplitude is applied on the same plane. If mean stresses are applied, the material constant A is considered to diminish moreover in proportion to the normal mean stress on the plane on which maximum shearing stress by mean stresses is applied. Then the new criterion is expressed as follows,

$$\tau_{\pi/4}=A-B\sigma_{\pi/4}-C\sigma_{\pi/4\cdot m} \tag{14}$$

where, $\tau_{\pi/4}$ and $\sigma_{\pi/4}$ are the shearing stress amplitude and the normal stress amplitude on the plane of maximum shearing stress respectively, and $\sigma_{\pi/4\cdot m}$ is the normal mean stress on the plane of maximum shearing stress by mean stresses. Eq. (14) is expressed by principal stresses as Eq. (15).

$$\frac{1}{2}(\sigma_1-\sigma_3)=A-B\frac{1}{2}(\sigma_1+\sigma_3)-C\frac{1}{2}(\sigma_{m1}+\sigma_{m3}) \tag{15}$$

where, σ_1 and σ_3 are the principal stress amplitudes, σ_{m1} and σ_{m3} are the mean principal stresses, and A, B and C are material constants. Eq. (15) is transformed for the case of bending and torsion. Denoting bending stress amplitude by σ_a, bending mean stress by σ_m, torsional stress amplitude by τ_a, torsional mean stress by τ_m,

$$\left.\begin{aligned}\sigma_1&=\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}, & \sigma_3&=\frac{1}{2}\sigma_a-\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}\\ \sigma_{m1}&=\frac{1}{2}\sigma_m+\frac{1}{2}\sqrt{\sigma_m^2+4\tau_m^2}, & \sigma_{m3}&=\frac{1}{2}\sigma_m-\frac{1}{2}\sqrt{\sigma_m^2+4\tau_m^2}\end{aligned}\right\} \tag{16}$$

Putting Eq. (16) in Eq. (15), Eq. (17) is obtained.

$$\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}=A-\frac{B}{2}\sigma_a-\frac{C}{2}\sigma_m \tag{17}$$

Constants A, B and C are determined applying known strength values. Applying

Eq. (17) for completely reversed torsion and putting $\sigma_a=\sigma_m=0$, $\tau_a=\tau_w$, we get

$$A=\tau_w \tag{18}$$

Applying Eq. (17) for completely reversed bending, and putting $\tau_a=\sigma_m=0$, $\sigma_a=\sigma_{wb}$ in Eq. (17), we get,

$$B=\frac{2\tau_w}{\sigma_{wb}}-1 \tag{19}$$

Applying Eq. (17) for pulsating bending, denoting pulsating bending fatigue limit $=\sigma_{up}$, and putting $\tau_a=0$, $\sigma_a=\sigma_m=\sigma_{up}/2$, $\sigma_{up}/(2\sigma_{wb})=k_1$ in Ea. (17), we get

$$C=\frac{2\tau_w}{\sigma_{wb}}\frac{1-k_1}{k_1} \tag{20}$$

Putting Eqs. (18)～(20) in Eq. (17), we get,

$$\left.\begin{array}{l} \left(\frac{\tau_a}{\tau_w}\right)^2\left(\frac{1}{p^2}\right)+\left(\frac{\sigma_a}{\sigma_{wb}}\right)^2\left\{\frac{(\sigma_{wb}/\tau_w)-1}{p^2}\right\}+\left(\frac{\sigma_a}{\sigma_{wb}}\right)\left\{\frac{2-(\sigma_{wb}/\tau_w)}{p}\right\}=1 \\ \text{where,} \\ p=1-\frac{1-k_1}{k_1}m_1,\ k_1=\frac{\sigma_{up}}{2\sigma_{wb}},\ m_1=\frac{\sigma_m}{\sigma_{wb}} \end{array}\right\} \tag{21}$$

If mean stresses are not applied, then $m_1=0$, $p=1$, and in this case, Eq. (21) coincides with Gough's Eq. (10). Then Eq. (21) is an extended criterion of modified maximum principal strain criterion, internal friction criterion and Gough's empirical formula for the case with mean stresses.

4. Comparison of the new criterion with experimental results

Comparison of Eq. (21) by author's criterion with the Gough's experimental results carried out on Ni-Cr-Mo-Va steel under combined static and repeated bending and torsion is made in Fig. 1 and Fig. 2. In Fig. 1, a mean stress is applied for bending, but in Fig. 2, mean stresses are applied for both bending and torsion. The value $k_1=\sigma_{up}/(2\sigma_{wb})=0.91$ is obtained by the Gough's experiment. In both Figs., the ratio of torsional stress amplitude and rotating bending fatigue limit is taken in ordinate, and the ratio of bending stress amplitude and rotating bending fatigue limit is taken in abscissa. In these Figs., Eq. (21) is shown by full lines, and these full lines pass through the middle of the experimental points.

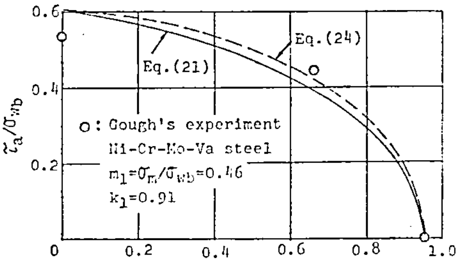

Fig. 1 Eatigue strength under combined bending and torsion with a mean stress (1)

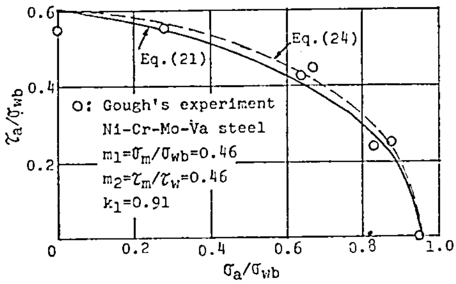

Fig. 2 Fatigue strength under combined bending and torsion with mean stresses (2)

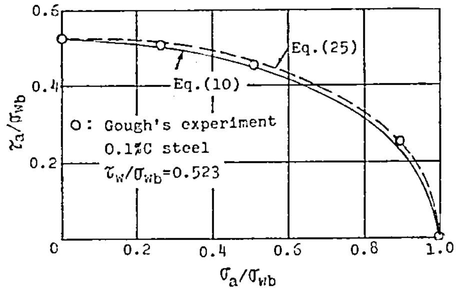

Fig. 3 Fatigue strength under combined bending and torsion without mean stresses (1)

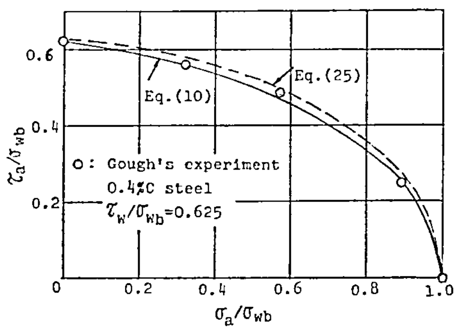

Fig. 4 Fatigue strength under combined bending and torsion without mean stresses (2)

In Figs. 3 and 4, Gough's experimental results on 0.1% carbon steel and on 0.4 % carbon steel under completely reversed bending and torsion were compared with author's criterion expressed with full line. Author's equation in this case coincides with Gough's empirical formula (10). Eq. (10) coincides well with experimental results.

5. Consideration

On the criteria of combined bending and torsion with mean stresses, we can mention Findley's criterion, Sines's criterion and Kakuno-Kawada's criterion. The difference between these three criteria and present author's criterion will be discussed here.

5-1 In Findley's criterion expressed by Eq. (4), the effect of the normal stress on the strength is considered to increase in proportion to the maximum normal stress $\sigma_{\theta M}$ which is the sum of mean stress and stress amplitude, but in author's criterion, expressed by Eq. (15), the effect of mean stress and stress amplitude is considered not the same. As the result, Eq. (21) which is easy to use for designers is derived from author's criterion, although more complicated equation is derived from Findley's criterion.

5-2 Sines's criterion[6] is expressed by the next equation.

$$\tau_{oct}=A-C\sigma_{oct\cdot m} \tag{22}$$

where τ_{oct}=octahedral shearing stress amplitude, $\sigma_{oct\cdot m}$=octahedral normal mean stress, A and C are material constants. Because Sines's criterion is octahedral shearing stress criterion if mean stresses are not applied, and then the ratio of torsional fatigue limit and bending fatigue limit should be 0.577. The experimental results of this ratio τ_w/σ_{wb} are known to scatter around the value 0.577 and author's criterion conforms better with experimental results than Sines's criterion.

5-3 Kakuno and Kawada's criterion[7] is such a modified Sines's criterion as it conforms better with experimental results. Fatigue failure is considered to occur when octahedral shearing stress amplitude τ_{oct} attains a material constant A, but the material constant A is diminished in proportion to the octahedral normal stress amplitude σ_{oct}, and if mean stresses are applied, the material constant A is moreover diminished in proportion to the octahedral normal mean stress $\sigma_{oct\cdot m}$. Then the criterion is written as follows,

$$\tau_{oct}=A-B\sigma_{oct}-C\sigma_{oct\cdot m} \tag{23}$$

where, B and C=material constants.

Expressing Eq. (23) with principal stresses, and moreover expressing the equation for the case of combined bending and torsion, we get Eq. (24) which resembles to Eq. (21).

$$\left.\begin{aligned}&\left(\frac{\tau_a}{\tau_w}\right)^2\left(\frac{1}{p^2}\right)+\left(\frac{\sigma_a}{\sigma_{wb}}\right)^2\left\{\frac{\frac{2}{\sqrt{3}}\left(\frac{\sigma_{wb}}{\tau_w}\right)-1}{p^2}\right\}+\left(\frac{\sigma_a}{\sigma_{wb}}\right)\left\{\frac{2-\frac{2}{\sqrt{3}}\left(\frac{\sigma_{wb}}{\tau_w}\right)}{p}\right\}=1\\&\text{where}\\&\qquad p=1-\frac{1-k_1}{k_1}m_1,\quad k_1=\frac{\sigma_{up}}{2\sigma_{wb}},\quad m_1=\frac{\sigma_m}{\sigma_{wb}}\end{aligned}\right\} \tag{24}$$

When mean stresses are not applied, $m_1=0$ and $p=1$, and Eq. (24) becomes Eq. (25).

$$\left(\frac{\tau_a}{\tau_w}\right)^2+\left(\frac{\sigma_a}{\sigma_{wb}}\right)^2\left\{\frac{2}{\sqrt{3}}\left(\frac{\sigma_{wb}}{\tau_w}\right)-1\right\}+\left(\frac{\sigma_a}{\sigma_{wb}}\right)\left\{2-\frac{2}{\sqrt{3}}\left(\frac{\sigma_{wb}}{\tau_w}\right)\right\}=1 \tag{25}$$

Eq. (24) and Eq. (25) are also used easily by designers. To compare Kakuno-Kawada's criterion with Gough's experimental results, Eq. (24) is expressed by dotted lines in Figs. 1 and 2, and Eq. (25) by dotted lines in Figs. 3 and 4. From these figures, next conclusions are derived. (1) The difference between author's criterion and Kakuno-Kawada's criterion is small in comparison with the scatter of experimental points. (2) Both author's criterion and Kakuno-Kawada's criterion are in accordance with experimental results in the same degree. (3) Though intermediate principal stress σ_2 is not included in author's criterion, as is shown in Eq. (15), all three principal stresses are included in Kakuno-Kawada's criterion as is shown in Eq. (23). Therefore, Kakuno-Kawada's criterion is considered better when triaxial stresses are applied. However, in the case of combined bending and torsion, intermediate principal stress $\sigma_2=0$, and then both criteria are equivalent. (4) Author's criterion is significant, because it includes several criteria stated in sections 2-1～2-6, as its special cases. (5) Author's criterion is considered to coexist with Kakuno-Kawada's criterion in the present state.

6. Conclusion

A new criterion of fatigue strength of a round bar subjected to combined static and repeated bending and torsion is proposed. The criterion is that, internal friction criterion considered on the plane of maximum shearing stress is extended to the case with mean stresses. The criterion coincides well with Gough's experimental results. Obtained design formula is simple, and the criterion includes several criteria already published as its special cases in the case of combined bending and torsion.

Author's criterion is considered to coexist with Kakuno-Kawada's criterion which is internal friction criterion considered on the octahedral plane and extended to the case with mean stresses.

Literature

1) Nishihara, T. and Endo, K. Trans. Japan Soc. Mech. Engrs., **17** (1951) 146.
2) Nakanishi, F., Trans. Japan Soc. Mech. Engrs. **18** (1952) 104, 110, 114, 118.
3) Findley, W. F., Trans. ASME, Ser. B, **81** (1959) 301.
4) Matake, T., Trans. Japan Soc. Mech. Engrs. **42** (1976) 1947.
5) Gough, H. J., Proc. Inst. Mech. Engs. **160** (1949) 417.
6) Sines, G., NACA Tech. Note, 3495 (1955).
7) Kakuno, H. and Kawada, Y., Fatigue of Engineering Materials and Structures, **2** (1979) 229, Pergamon Press.