AN EXACT SEQUENCE INDUCED BY THE GROUP SCHEME DEFORMING $\mathbb{G}_a$ TO $\mathbb{G}_m$

MICHIO AMANO

ABSTRACT. Let $\mathcal{G}^{(\lambda)}$ be a group scheme which deforms the additive group scheme $\mathbb{G}_a$ to the multiplicative group scheme $\mathbb{G}_m$. We consider the short exact sequence

$$0 \longrightarrow N_l \xrightarrow{\iota} \widehat{\mathcal{G}}^{(\lambda)} \xrightarrow{\psi^{(l)}} \widehat{\mathcal{G}}^{(\lambda^{p^l})} \longrightarrow 0$$

induced by a Frobenius-type surjective homomorphism $\psi^{(l)}$, where $\widehat{\mathcal{G}}^{(\lambda)}$ is the formal completion of $\mathcal{G}^{(\lambda)}$ along the zero section. In this case, we show that the sequence

$$0 \longrightarrow \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_m) \xrightarrow{(\psi^{(l)})^*} \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_m) \xrightarrow{(\iota)^*} \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_m) \longrightarrow 0$$

is exact in positive characteristic p. After this, the injection

$$\mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_m) \hookrightarrow \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_m) \quad (\text{resp. } H_0^2(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_m) \hookrightarrow H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_m))$$

is obtained.

1. Introduction

Throughout this paper, we denote by p a prime number. Let A be a $\mathbb{Z}_{(p)}$-algebra and λ a suitable element of A, where $\mathbb{Z}_{(p)}$ is a localization of rational integers $\mathbb{Z}$ at p. The group scheme $\mathcal{G}^{(\lambda)} = \mathrm{Spec}\, A[T, 1/(1+\lambda T)]$ which deforms the additive group scheme $\mathbb{G}_a$ to the multiplicative group scheme $\mathbb{G}_m$ has been constructed by [7]. Then $\mathcal{G}^{(\lambda)}$ has been studied in detail by [4] for the purpose of unifying Kummer theory and Artin-Schreier theory.

Let l be a positive integer. For any $1 \le i \le p^l$, there exist k_i uniquely such that $p^{k_i} \le i < p^{k_i+1}$. Note that $0 \le k_i \le l$ for any $i = 1, 2, \ldots, p^l$. For each integer $0 \le k \le l-1$, we take $\lambda, \nu_k \in A$ such that $p^{l-k}\lambda^{p^k} = \nu_k \lambda^{p^l}$. Put $\psi^{(l)}(X) := \sum_{i=1}^{p^l-1} a_i X^i + X^{p^l}$, where $a_i = \binom{p^l}{i} p^{-(l-k_i)} \lambda^{r_i} \nu_{k_i}$. Then $\psi^{(l)}(X)$ satisfies $\lambda^{p^l}\psi^{(l)}(X) = (1+\lambda X)^{p^l} - 1$. For a group scheme G, we denote by $\widehat{G}$ the formal completion along the zero section. For an endomorphism

$$\varphi : \widehat{\mathbb{G}}_{m,A} \to \widehat{\mathbb{G}}_{m,A};\ t \mapsto t^{p^l},$$

we obtain the commutative diagram

$$\begin{array}{ccc}
\widehat{\mathcal{G}}^{(\lambda)} & \xrightarrow{\alpha^{(\lambda)}} & \widehat{\mathbb{G}}_{m,A} \\
{\scriptstyle \psi^{(l)}}\downarrow & & \downarrow{\scriptstyle \varphi} \\
\widehat{\mathcal{G}}^{(\lambda^{p^l})} & \xrightarrow{\alpha^{(\lambda^{p^l})}} & \widehat{\mathbb{G}}_{m,A},
\end{array}$$

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where $\alpha^{(\lambda)}(x) = 1 + \lambda x$. Then

$$\psi^{(l)} : \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathcal{G}}^{(\lambda^{p^l})};\ x \mapsto \psi^{(l)}(x)$$

is a well-defined surjective homomorphism with $N_l := \mathrm{Ker}(\psi^{(l)})$. Note that N_l is a finite group scheme of order p^l, under the assumption that A is an $\mathbb{F}_p$-algebra. Then the author has determined the Cartier dual of N_l in [1]. The following short exact sequence is induced by $\psi^{(l)}$:

$$0 \longrightarrow N_l \xrightarrow{\ \iota\ } \widehat{\mathcal{G}}^{(\lambda)} \xrightarrow{\psi^{(l)}} \widehat{\mathcal{G}}^{(\lambda^{p^l})} \longrightarrow 0, \tag{1}$$

where ι is a canonical inclusion. The exact sequence (1) deduces the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) &\xrightarrow{(\psi^{(l)})^*} \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\iota)^*} \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A}) \\ \xrightarrow{\ \partial\ } \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) &\xrightarrow{(\psi^{(l)})^*} \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \longrightarrow \cdots. \end{aligned}$$

Since the image of the boundary map ∂ is given by direct product of formal schemes, we can replace $\mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A})$ (resp. $\mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A})$) with $H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A})$ (resp. $H_0^2(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A})$). Therefore the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) &\xrightarrow{(\psi^{(l)})^*} \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\iota)^*} \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A}) \\ \xrightarrow{\ \partial\ } H_0^2(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) &\xrightarrow{(\psi^{(l)})^*} H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \longrightarrow \cdots \end{aligned}$$

is obtained. Then the main result of this paper is:

Theorem 1. *Let A be an $\mathbb{F}_p$-algebra. With these notations, we have the short exact sequence:*

$$0 \longrightarrow \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_m) \xrightarrow{(\psi^{(l)})^*} \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_m) \xrightarrow{(\iota)^*} \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_m) \longrightarrow 0.$$

Theorem 1 is obtained by showing that

$$(\iota)^* : \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \to \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A})$$

is a surjective homomorphism (for the proof, see Section 4). By using Theorem 1, the equality $\mathrm{Ker}(\psi^{(l)})^* = 1$ is immediately shown. Then we have:

Corollary 1. *Let A be an $\mathbb{F}_p$-algebra. With the above assumptions, there exists the injection*

$$(\psi^{(l)})^* : \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_m) \hookrightarrow \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_m) \quad (\textit{resp. } (\psi^{(l)})^* : H_0^2(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_m) \hookrightarrow H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_m)).$$

The contents of this paper are as follows. The next two sections are devoted to recalling the definitions and some properties of the Witt scheme and of the group scheme which deforms $\mathbb{G}_a$ to $\mathbb{G}_m$. In Section 4 we give the proof of Theorem 1.

Throughout this paper, we use the following notations:

$\mathbb{G}_{a,A}$: additive group scheme over A,
$\mathbb{G}_{m,A}$: multiplicative group scheme over A,
$\widehat{\mathbb{G}}_{m,A}$: multiplicative formal group scheme over A,
$W_{n,A}$: group scheme of Witt vectors of length n over A,
W_A : group scheme of Witt vectors over A,
F : Frobenius endomorphism of W_A,
V : Verschiebung endomorphism of W_A,

$[\lambda]$: Teichmüller lifting $(\lambda, 0, 0, \ldots) \in W(A)$ of $\lambda \in A$,
$F^{(\lambda)}$: $= F - [\lambda^{p-1}]$,
$T_{\boldsymbol{a}}$: homomorphism decided by $\boldsymbol{a} \in W(A)$ (recalled in Section 2),
$W(A)^{F^{(\lambda)}}$: $= \mathrm{Ker}[F^{(\lambda)} : W(A) \to W(A)]$,
$W(A)/F^{(\lambda)}$: $= \mathrm{Coker}[F^{(\lambda)} : W(A) \to W(A)]$,
$\mathrm{Ext}^1(G, H)$: isomorphism classes of extensions of G by H,
$H_0^2(G, H)$: Hochschild cohomology of G with coefficients in H.

2. Witt vectors

In this short section, we recall the necessary facts on Witt vectors in this paper. For details, see [2, Chap. V] or [3, Chap. III].

2.1. **Definition of Witt vectors.** Let $\mathbb{X} = (X_0, X_1, \ldots)$ be a sequence of variables. For each $n \geq 0$, we denote by $\Phi_n(\mathbb{X}) = \Phi_n(X_0, X_1, \ldots, X_n)$ the Witt polynomial

$$\Phi_n(\mathbb{X}) = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n$$

in $\mathbb{Z}[\mathbb{X}] = \mathbb{Z}[X_0, X_1, \ldots]$. Let $W_{\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z}[\mathbb{X}]$ be an ∞-dimensional affine space over $\mathbb{Z}$. The phantom map Φ is defined by

$$\Phi : W_{\mathbb{Z}} \to \mathbb{A}_{\mathbb{Z}}^{\infty};\ \boldsymbol{x} \mapsto (\Phi_0(\boldsymbol{x}), \Phi_1(\boldsymbol{x}), \ldots),$$

where $\mathbb{A}_{\mathbb{Z}}^{\infty}$ is the usual ∞-dimensional affine space over $\mathbb{Z}$. The scheme $\mathbb{A}_{\mathbb{Z}}^{\infty}$ has a natural ring scheme structure. It is known that $W_{\mathbb{Z}}$ has a unique commutative ring scheme structure over $\mathbb{Z}$ such that the phantom map Φ is a homomorphism of commutative ring schemes over $\mathbb{Z}$. Then A-valued points $W_{\mathbb{Z}}(A)$ are called Witt vectors over A.

2.2. **Some morphisms of Witt vectors.** We define a morphism $F : W(A) \to W(A)$ by

$$\Phi_i(F(\boldsymbol{x})) = \Phi_{i+1}(\boldsymbol{x})$$

for $\boldsymbol{x} \in W(A)$. If A is an $\mathbb{F}_p$-algebra, F is nothing but the usual Frobenius endomorphism. Let $[\lambda]$ be the Teichmüller lifting $[\lambda] = (\lambda, 0, 0, \ldots) \in W(A)$ for $\lambda \in A$. Then we set the endomorphism $F^{(\lambda)} := F - [\lambda^{p-1}]$ of $W(A)$.

For $\boldsymbol{a} = (a_0, a_1, \ldots) \in W(A)$, we also define a morphism $T_{\boldsymbol{a}} : W(A) \to W(A)$ by

$$\Phi_n(T_{\boldsymbol{a}}(\boldsymbol{x})) = {a_0}^{p^n}\Phi_n(\boldsymbol{x}) + p{a_1}^{p^{n-1}}\Phi_{n-1}(\boldsymbol{x}) + \cdots + p^n a_n \Phi_0(\boldsymbol{x})$$

for $\boldsymbol{x} \in W(A)$ ([5, Chap.4, p.20]). Then the morphism $T_{\boldsymbol{a}}$ is called T-map.

3. The group scheme $\mathcal{G}^{(\lambda)}$ which deforms $\mathbb{G}_a$ to $\mathbb{G}_m$

In this short section, we recall necessary facts on the group scheme $\mathcal{G}^{(\lambda)}$ which deforms $\mathbb{G}_a$ to $\mathbb{G}_m$ for this paper.

3.1. **Definition of the group scheme $\mathcal{G}^{(\lambda)}$.** Let A be a ring and λ an element of A. Put $\mathcal{G}^{(\lambda)} := \mathrm{Spec}\,A[X, 1/(1 + \lambda X)]$. We define a morphism $\alpha^{(\lambda)}$ by

$$\alpha^{(\lambda)} : \mathcal{G}^{(\lambda)} \to \mathbb{G}_{m,A};\ x \mapsto 1 + \lambda x.$$

It is known that $\mathcal{G}^{(\lambda)}$ has a unique commutative group scheme structure such that $\alpha^{(\lambda)}$ is a group scheme homomorphism over A. Then the group scheme structure of $\mathcal{G}^{(\lambda)}$ is given by $x \cdot y = x + y + \lambda xy$. If λ is invertible in A, $\alpha^{(\lambda)}$ is an A-isomorphism. On the other hand, if $\lambda = 0$, $\mathcal{G}^{(\lambda)}$ is nothing but the additive group scheme $\mathbb{G}_{a,A}$.

3.2. **Deformed Artin-Hasse exponential series.** The Artin-Hasse exponential series $E_p(X)$ is given by

$$E_p(X) = \exp\left(\sum_{r \geq 0} \frac{X^{p^r}}{p^r}\right) \in \mathbb{Z}_{(p)}[[X]].$$

We define a formal power series $E_p(U,\Lambda;X)$ in $\mathbb{Q}[U,\Lambda][[X]]$ by

$$E_p(U,\Lambda;X)=(1+\Lambda X)^{\frac{U}{\Lambda}}\prod_{k=1}^{\infty}(1+\Lambda^{p^k}X^{p^k})^{\frac{1}{p^k}((\frac{U}{\Lambda})^{p^k}-(\frac{U}{\Lambda})^{p^{k-1}})}.$$

As in [6, Corollary 2.5.] or [5, Lemma 4.8.], we see that the formal power series $E_p(U,\Lambda;X)$ is integral over $\mathbb{Z}_{(p)}$. Note that $E_p(1,0;X)=E_p(X)$.

Let A be a $\mathbb{Z}_{(p)}$-algebra. For $\lambda\in A$ and $\boldsymbol{v}=(v_0,v_1,\ldots)\in W(A)$, we define a formal power series $E_p(\boldsymbol{v},\lambda;X)$ in $A[[X]]$ by

$$E_p(\boldsymbol{v},\lambda;X)=\prod_{k=0}^{\infty}E_p(v_k,\lambda^{p^k};X^{p^k})=(1+\lambda X)^{\frac{v_0}{\lambda}}\prod_{k=1}^{\infty}(1+\lambda^{p^k}X^{p^k})^{\frac{1}{p^k\lambda^{p^k}}\Phi_{k-1}(F^{(\lambda)}(\boldsymbol{v}))}.$$

Moreover we define a formal power series $F_p(\boldsymbol{v},\lambda;X,Y)$ as follows:

$$F_p(\boldsymbol{v},\lambda;X,Y)=\prod_{k=1}^{\infty}\left(\frac{(1+\lambda^{p^k}X^{p^k})(1+\lambda^{p^k}Y^{p^k})}{1+\lambda^{p^k}(X+Y+\lambda XY)^{p^k}}\right)^{\frac{1}{p^k\lambda^{p^k}}\Phi_{k-1}(\boldsymbol{v})}.$$

As in [6, Lemma 2.16.] or [5, Lemma 4.9.], we see that the formal power series $F_p(\boldsymbol{v},\lambda;X,Y)$ is integral over $\mathbb{Z}_{(p)}$.

4. Proof of Theorem 1

In this section we give a proof of Theorem 1.

Suppose that A is an $\mathbb{F}_p$-algebra. Let l be a positive integer. For any $1\le i\le p^l$, there exist k_i uniquely such that $p^{k_i}\le i<p^{k_i+1}$ and $i=p^{k_i}+r_i$. Note that $0\le k_i\le l$ for any $i=1,2,\cdots,p^l$. For each integer $0\le k\le l-1$, we take $\lambda,\nu_k\in A$ such that $p^{l-k}\lambda^{p^k}=\nu_k\lambda^{p^l}$. Let $\mathcal{G}^{(\lambda)}$ be the group scheme defined in Subsection 3.1 and $\widehat{\mathcal{G}}^{(\lambda)}$ the formal completion of $\mathcal{G}^{(\lambda)}$ along the zero section. We put

$$\psi^{(l)}(X):=\sum_{i=1}^{p^l-1}a_iX^i+X^{p^l}\in A[X]$$

where

$$a_i=\binom{p^l}{i}p^{-(l-k_i)}\lambda^{r_i}\nu_{k_i}\in A.$$

Then $\psi^{(l)}(X)$ satisfies the equality

$$\lambda^{p^l}\psi^{(l)}(X)=(1+\lambda X)^{p^l}-1.$$

Hence

$$\psi^{(l)}:\widehat{\mathcal{G}}^{(\lambda)}\to\widehat{\mathcal{G}}^{(\lambda^{p^l})};\ x\mapsto\psi^{(l)}(x)$$

is a well-defined surjective homomorphism such that the following diagram is commutative:

$$\begin{array}{ccc}
\widehat{\mathcal{G}}^{(\lambda)} & \xrightarrow{\alpha^{(\lambda)}} & \widehat{\mathbb{G}}_{m,A}\\
{\scriptstyle\psi^{(l)}}\downarrow & & \downarrow{\scriptstyle t\mapsto t^{p^l}}\\
\widehat{\mathcal{G}}^{(\lambda^{p^l})} & \xrightarrow{\alpha^{(\lambda^{p^l})}} & \widehat{\mathbb{G}}_{m,A}.
\end{array}$$

The short exact sequence

$$0\longrightarrow N_l\xrightarrow{\iota}\widehat{\mathcal{G}}^{(\lambda)}\xrightarrow{\psi^{(l)}}\widehat{\mathcal{G}}^{(\lambda^{p^l})}\longrightarrow 0 \tag{2}$$

is induced by $\psi^{(l)}$, where ι is a canonical inclusion. The short exact sequence (2) determines the long exact sequence

$$0\longrightarrow\mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})},\widehat{\mathbb{G}}_{m,A})\xrightarrow{(\psi^{(l)})^*}\mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)},\widehat{\mathbb{G}}_{m,A})\xrightarrow{(\iota)^*}\mathrm{Hom}(N_l,\widehat{\mathbb{G}}_{m,A})$$

$$\xrightarrow{\partial} \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\psi^{(l)})^*} \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \longrightarrow \quad \cdots .$$

Since the image of the boundary map ∂ is given by direct product of formal schemes, we can replace $\mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A})$ (resp. $\mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A})$) with $H^2_0(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A})$ (resp. $H^2_0(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A})$). Hence, we get the long exact sequence

$$\begin{aligned} &0 \longrightarrow \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\psi^{(l)})^*} \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\iota)^*} \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A}) \\ &\xrightarrow{\partial} H^2_0(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) \xrightarrow{(\psi^{(l)})^*} H^2_0(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \longrightarrow \quad \cdots . \end{aligned} \tag{3}$$

Here $H^2_0(G, H)$ denotes the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in H for formal group schemes G and H ([2, Chap. II.3 and Chap. III.6]).

On the other hand, as in [6, Theorem 2.19.1] or the case $n = 1$ of [5, Theorem 5.1.], the following morphisms are isomorphic:

$$\begin{aligned} \mathrm{Ker}[F^{(\lambda)} : W(A) \to W(A)] &\to \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}); \\ \boldsymbol{v} &\mapsto E_p(\boldsymbol{v}, \lambda; x) \end{aligned} \tag{4}$$

$$\begin{aligned} \mathrm{Coker}[F^{(\lambda)} : W(A) \to W(A)] &\to H^2_0(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}); \\ \boldsymbol{w} &\mapsto F_p(\boldsymbol{w}, \lambda; x, y). \end{aligned} \tag{5}$$

Here we choose $\boldsymbol{a} \in W(A)$ such that $T_{\boldsymbol{a}}([\lambda^{p^l}]) = p^l[\lambda]$, where $T_{\boldsymbol{a}}([\lambda^{p^l}]) = (\lambda^{p^l} a_0, \lambda^{p^l} a_1, \ldots)$ for $\boldsymbol{a} = (a_0, a_1, \ldots)$. Then we set $W_{l,A}(B) := \mathrm{Coker}[T_{\boldsymbol{a}} : W_A(B) \to W_A(B)]$ for any A-algebra B. Then $W_{l,A}$ is a sheaf on A-algebras. In fact, for any A-algebra B, we have the following exact sequence on the fppf site:

$$0 \longrightarrow W_A(B) \xrightarrow{T_a} W^2_A(B) \longrightarrow \widetilde{W^2_A/T_0}(B) \longrightarrow H^1(\mathrm{Spec}(B), W^2_A),$$

where $\widetilde{W_A/T_{\boldsymbol{a}}}$ is the fppf-sheafification of $W_A/T_{\boldsymbol{a}}$. Since W_A is a quasi-coherent sheaf, the cohomology is trivial. This shows

$$\widetilde{W_A/T_{\boldsymbol{a}}}(B) \simeq W_A(B)/T_{\boldsymbol{a}}(B) = W_{l,A}(B).$$

We consider the diagram

$$\begin{array}{ccc} W(A) & \xrightarrow{\pi} & W_l(A) \\ {\scriptstyle F^{(\lambda)}}\downarrow & & \downarrow{\scriptstyle \overline{F^{(\lambda)}}} \\ W(A) & \xrightarrow{\pi} & W_l(A), \end{array}$$

where π is a natural projection and $\overline{F^{(\lambda)}}$ is defined by $\overline{F^{(\lambda)}}(\overline{\boldsymbol{x}}) := \overline{F^{(\lambda)}(\boldsymbol{x})}$. Then the homomorphism $\overline{F^{(\lambda)}}$ is well-defined and the above diagram is commutative, since $F^{(\lambda)} \circ T_{\boldsymbol{a}}(\boldsymbol{x}) = T_{\boldsymbol{a}} \circ F^{(\lambda^{p^l})}(\boldsymbol{x})$ for $\boldsymbol{x} \in W(A)$. Note that $T_{\boldsymbol{a}} = V^l$ holds under the assumption that A is an $\mathbb{F}_p$-algebra. Hence the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(A) & \xrightarrow{T_a} & W(A) & \xrightarrow{\pi} & W_l(A) & \longrightarrow & 0 \\ & & {\scriptstyle F^{(\lambda^{p^l})}}\downarrow & & {\scriptstyle F^{(\lambda)}}\downarrow & & {\scriptstyle \overline{F^{(\lambda)}}}\downarrow & & \\ 0 & \longrightarrow & W(A) & \xrightarrow{T_a} & W(A) & \xrightarrow{\pi} & W_l(A) & \longrightarrow & 0, \end{array}$$

is commutative. By using the snake lemma for this diagram, we have the exact sequence

$$\begin{aligned} &0 \longrightarrow W(A)^{F^{(\lambda^{p^l})}} \xrightarrow{T_a} W(A)^{F^{(\lambda)}} \xrightarrow{\pi} W_l(A)^{F^{(\lambda)}} \\ &\xrightarrow{\partial} W(A)/F^{(\lambda^{p^l})} \xrightarrow{T_a} W(A)/F^{(\lambda)} \xrightarrow{\pi} W_l(A)/F^{(\lambda)} \longrightarrow 0. \end{aligned} \tag{6}$$

Now, by combining the exact sequences (3), (6) and the isomorphisms (4), (5), we have the following diagram consisting of exact horizontal lines and vertical isomorphisms except ϕ:

$$
\begin{array}{ccccc}
\mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A}) \\
\phi_1 \uparrow & & \phi_2 \uparrow & & \phi \uparrow \\
W(A)^{F^{(\lambda^{p^l})}} & \xrightarrow{T_a} & W(A)^{F^{(\lambda)}} & \xrightarrow{\pi} & W_l(A)^{F^{(\lambda)}} \\
 & \xrightarrow{\partial} & H^2_0(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbb{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & H^2_0(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \\
 & & \phi_3 \uparrow & & \phi_4 \uparrow \\
 & \xrightarrow{\partial} & W(A)/F^{(\lambda^{p^l})} & \xrightarrow{T_a} & W(A)/F^{(\lambda)},
\end{array}
\tag{7}
$$

where $W_l(A)^{F^{(\lambda)}} := \mathrm{Ker}[\overline{F^{(\lambda)}} : W_l(A) \to W_l(A)]$ and ϕ is the following homomorphism induced from the exact sequence (2) and the isomorphism (4):

$$\phi : W_l(A)^{F^{(\lambda)}} \to \mathrm{Hom}(N_l, \widehat{\mathbb{G}}_{m,A});\ \overline{\boldsymbol{x}} \mapsto E_p(\overline{\boldsymbol{x}}, \lambda; x, y) := E_p(\boldsymbol{x}, \lambda; x, y).$$

The well-definedness of ϕ has already been shown in [1, Lemma 1, p.123]. Moreover it has already been proved in [1, Section 4] that ϕ is an isomorphism. Hence (7) is a commutative diagram consisting of exact horizontal lines and vertical isomorphisms. Therefore $(\iota)^*$ to be surjective and π to be surjective are equivalent. Then, if

$$\pi : W(A)^{F^{(\lambda)}} \to W_l(A)^{F^{(\lambda)}};\ \boldsymbol{x} \mapsto \pi(\boldsymbol{x}) = \overline{\boldsymbol{x}} \tag{8}$$

is shown to be surjective, Theorem 1 is proved.

Proposition 1. *The homomorphism* (8) *is surjective.*

Proof Take any $\overline{\boldsymbol{x}} \in W_l(A)^{F^{(\lambda)}}$. This means $F^{(\lambda)}(\boldsymbol{x}) \in \mathrm{Im}(T_{\boldsymbol{a}})(A)$. Then there exists $\boldsymbol{z} \in W(A)$ such that $F^{(\lambda)}(\boldsymbol{x}) = T_{\boldsymbol{a}}(\boldsymbol{z})$. Put $\boldsymbol{w} := F^{(\lambda)}(\boldsymbol{x}) = T_{\boldsymbol{a}}(\boldsymbol{z})$, where $\boldsymbol{w} = (w_0, w_1, \ldots) \in W(A)$. We can calculate $\boldsymbol{w} \in W(A)$ by using the phantom map. In our case, since $T_{\boldsymbol{a}} = V^l$, the equalities

$$w_0 = w_1 = \cdots = w_{l-1} = 0$$

holds. Set $\boldsymbol{x} := (x_0, x_1, \ldots) \in W(A)$. For $\Phi_0(\boldsymbol{w}) = \Phi_0(F^{(\lambda)}(\boldsymbol{x}))$, we have the equality

$$x_0^p - \lambda^{p-1} x_0 \equiv 0 \pmod p.$$

Inductively,

$$x_i^{p^{i+1}} - \lambda^{p^i(p-1)} x_i^{p^i} \equiv 0 \pmod p \tag{9}$$

are obtained for $0 \le i \le l-1$. Next, $\Phi_l(\boldsymbol{w}) = \Phi_l(F^{(\lambda)}(\boldsymbol{x}))$ means

$$\begin{aligned} & w_0^{p^l} + p w_1^{p^{l-1}} + \cdots + p^l w_l \\ & = \left(x_0^{p^{l+1}} + p x_1^{p^l} + \cdots + p^l x_l^p + p^{l+1} x_{l+1} \right) - \lambda^{p^l(p-1)} \left(x_0^{p^l} + p x_1^{p^{l-1}} + \cdots + p^l x_l \right). \end{aligned}$$

This leads to

$$\begin{aligned} w_l &= \frac{1}{p^l} \left(x_0^{p^{l+1}} - \lambda^{p^l(p-1)} x_0^{p^l} \right) + \frac{1}{p^{l-1}} \left(x_1^{p^l} - \lambda^{p^l(p-1)} x_1^{p^{l-1}} \right) + \cdots + \left(x_l^p - \lambda^{p^l(p-1)} x_l \right) + p x_{l+1} \\ &\equiv x_l^p - \lambda^{p^l(p-1)} x_l \pmod p \end{aligned}$$

by using the equalities (9). Inductively, we obtain

$$w_{l+k} = x_{l+k}^{p^{k+1}} - \lambda^{p^{l+k}(p-1)} x_{l+k}^{p^k} \pmod p$$

for any integer $k \ge 0$. These imply that $\boldsymbol{w} = T_{\boldsymbol{a}} \circ F^{(\lambda^{p^l})}(\boldsymbol{x}_l)$, where $\boldsymbol{x}_l = (x_l, x_{l+1}, \ldots)$. Then, since

$$\boldsymbol{w} = T_{\boldsymbol{a}} \circ F^{(\lambda^{p^l})}(\boldsymbol{x}_l) = F^{(\lambda)} \circ T_{\boldsymbol{a}}(\boldsymbol{x}_l) = F^{(\lambda)}(\boldsymbol{x}),$$

$F^{(\lambda)}(T_{\boldsymbol{a}}(\boldsymbol{x}_l) - \boldsymbol{x}) = 0$ holds. Therefore we have $T_{\boldsymbol{a}}(\boldsymbol{x}_l) - \boldsymbol{x} \in W(A)^{F^{(\lambda)}}$ such that

$$\pi(T_{\boldsymbol{a}}(\boldsymbol{x}_l) - \boldsymbol{x}) = \overline{\boldsymbol{x}} \in W_l(A)^{F^{(\lambda)}}.$$

This means that (8) is surjective. □

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School of Education, Meisei University, 2-1-1 Hodokubo Hino, Tokyo 191-8506, Japan
Email address: michio.amano@meisei-u.ac.jp